Reduced-dimension Models in Nonlinear Finite Element Dynamics of Continuous Media

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Summary

Model reduction is a means of arriving at simplified dynamical descriptions of the behavior of a complex system that may be hard to model directly. Instead of attempting to model the overall behavior of such systems by considering all constituent degrees of freedom, a simplified model is built with few degrees of freedom, but capable of modeling the observed behavior at a particular scale of interest using simulation or experimental data. We provide a consistent method of constructing reduced-order models of mechanical systems which preserves the Lagrangian structure of the original system. The optimal representation is derived by the method of proper orthogonal decomposition, which has been variously termed in the literature as factor analysis, principal components analysis, and Karhunen-Love expansion (KLE). The model reduction procedure is implemented for three-dimensional finite-element models of solid dynamics. We show how using the standard Newmark implicit integrator significant savings are obtained in the computational costs, because the reduced model scales linearly in the number of degrees of freedom, and efficiently parallelizes.

Introduction

In general, finite element (FE) analysis of dynamic non-linear deformations of solids and structures can be rather expensive, especially in situations when one requires many repeated trials, as often happens in design and control applications. Explicit time discretization avoids storage of large matrices and repeated costly solution of systems of linear equations, but is limited by its conditional stability. Implicit time stepping algorithms require repeated solutions of large systems of linear equations, which consume considerable resources both in terms of the computation time and in terms of the data storage requirements.

The standard FE approximation basis is non-optimal in the sense that the coherence (large scale structures) in the displacement field is not exploited. In contrast, consider linear forced vibrations of solids: Decomposition of the displacement field into vibration modes is optimal in the sense that depending on the frequency content of the loading, a relatively small number of modes is sufficient to describe the motion for a given error tolerance.

The principal idea of dimensional model reduction is to find a small number of generalized coordinates in which to express the dynamics, ideally with some bounds on the truncation error. One way of applying this idea in the context of FE models is a transition from the collection of individual nodal basis functions to several linear combinations of the nodal basis functions (modes, or generalized coordinates). The question is then how to compute the amplitudes of the FE basis functions for the needed modes, and how to integrate the reduced dynamical system in time.

Recent results on model reduction of general Lagrangian systems were developed by Lall, Krysl and Marsden [3], and the paper of Krysl, Lall, and Marsden [2] focuses on methods for reduction of finite-element models.

Nonlinear Finite Element Dynamics

Finite element discretization of solid mechanics boundary value problems leads to equations of the form

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) = \boldsymbol{f}^{\text{ext}}(t) - \boldsymbol{f}^{\text{int}}(\boldsymbol{u}(t)) \,, \tag{1}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{u}$ is the vector of displacements, and $\boldsymbol{f}^{\text{int}}$, $\boldsymbol{f}^{\text{ext}}$ are the internal and external forces (the latter are simplicity assumed to be independent of $\boldsymbol{u}$).

Equation (1) represent a system of nonlinear ODE's, and are usually discretized with an explicit or implicit version of the Newmark algorithm, which is also our choice [1]:

$$\begin{aligned} \boldsymbol{M}\boldsymbol{a}_{t+\Delta t} &= \boldsymbol{f}^{\text{ext}}_{t+\Delta t} - \boldsymbol{f}^{\text{int}}_{t+\Delta t} \\ \boldsymbol{u}_{t+\Delta t} &= \boldsymbol{u}_t + \Delta t \boldsymbol{v}_t + \frac{\Delta t^2}{2}\left[(1-2\beta)\boldsymbol{a}_t + 2\beta \boldsymbol{a}_{t+\Delta t}\right] \\ \boldsymbol{v}_{t+\Delta t} &= \boldsymbol{v}_t + \Delta t\left[(1-\gamma)\boldsymbol{a}_t + \gamma \boldsymbol{a}_{t+\Delta t}\right] \end{aligned} \tag{2}$$

Here we deal with the implicit version only ($\gamma = 1/2$, $\beta \geq 1/4$); reduction of the explicit algorithm is discussed in Reference [2].

Local mode superposition

Model reduction has been in the past applied in the form of ***local mode superposition*** [4]. The incremental problem (2) is viewed as linear vibration superimposed onto the nonlinear overall motion. One proceeds by re-writing the equation of motion (1) in the form of a virtual work principle

$$\delta\boldsymbol{u}^T\left(\boldsymbol{M}\ddot{\boldsymbol{u}}_\tau + \boldsymbol{f}^{int}_\tau - \boldsymbol{f}^{ext}_\tau\right) = 0 \,, \tag{3}$$

which is assumed to hold for $\tau = t$ and $\tau = t + \Delta t$. Linearizing (1) at $\tau = t$ and assuming that (3) is satisfied by the linearized expression at $\tau = t + \Delta t$ leads to

$$\delta\boldsymbol{u}^T\left(\boldsymbol{M}\Delta\ddot{\boldsymbol{u}}_t + \boldsymbol{K_T}\Delta\boldsymbol{u}_t\right) = \delta\boldsymbol{u}\left(\boldsymbol{f}^{ext}_{t+\Delta t} - \boldsymbol{f}^{ext}_t\right) \,, \tag{4}$$

where $\boldsymbol{K_T}$ is the tangent stiffness matrix, and $\Delta\boldsymbol{u}$ is the displacement increment.

At this point, let us assume there exists an optimal Ritz basis $\boldsymbol{y}^j$. Then, one can use an expansion of both the displacement variation and the increment in terms of ***Ritz modes*** $\boldsymbol{y}^j$,

$$\delta\boldsymbol{u} = \boldsymbol{\Phi}\delta\boldsymbol{x} \,, \text{ and } \Delta\boldsymbol{u} = \boldsymbol{\Phi}\Delta\boldsymbol{x} \,, \tag{5}$$

where the ***modal matrix*** $\boldsymbol{\Phi}$ is a rectangular matrix with amplitudes of mode $\boldsymbol{y}^j$ in the jth column, and $\boldsymbol{x}$ is the vector of ***modal mixing coefficients***. Equation (4) is transformed by the modal decomposition into

$$\delta\boldsymbol{x}^T\left(\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}\Delta\ddot{\boldsymbol{x}}_t + \boldsymbol{\Phi}^T\boldsymbol{K_T}\boldsymbol{\Phi}\Delta\boldsymbol{x}_t\right) = \delta\boldsymbol{x}^T\boldsymbol{\Phi}^T\left(\boldsymbol{f}^{ext}_{t+\Delta t} - \boldsymbol{f}^{ext}_t\right) \,. \tag{6}$$

The classical local mode superposition takes advantage of the formal similarity of (4) to an equation of linear vibration, and proceeds by computing the modal decomposition of the linear eigenvibration problem.

However, such a basis is likely to be optimal only in a local sense, and hence needs to be recomputed quite often, which is expensive. Moreover, it does not into account the load distribution. To ameliorate both aspects, the eigenvibration basis is sometimes supplemented by higher-order derivatives of the solution or by so-called Ritz load-dependent vectors [2]. The scalability of these reduced procedures remains questionable.

Present Approach

Equations (2) are most often solved by some variant of the Newton iteration procedure. The displacement is incremented by the Newmark predictor, and then corrected by $\Delta\boldsymbol{u}^{(i)}$ (i denotes the iteration), which needs to be solved for from the system of linear equations

$$\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M}+\boldsymbol{K}_T^{(i)}\right)\Delta\boldsymbol{u}^{(i)}=\boldsymbol{M}\boldsymbol{a}_{t+\Delta t}^{(i)}-\boldsymbol{f}_{t+\Delta t}^{\mathrm{ext}(i)}+\boldsymbol{f}_{t+\Delta t}^{\mathrm{int}(i)}\ . \tag{7}$$

Reasoning which led to equation (6) may now be applied to transform (7) into

$$\boldsymbol{\Phi}^T\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M}+\boldsymbol{K}_T^{(i)}\right)\boldsymbol{\Phi}\Delta\boldsymbol{x}^{(i)}=\boldsymbol{\Phi}^T\left(\boldsymbol{M}\boldsymbol{a}_{t+\Delta t}^{(i)}-\boldsymbol{f}_{t+\Delta t}^{\mathrm{ext}(i)}+\boldsymbol{f}_{t+\Delta t}^{\mathrm{int}(i)}\right)\ . \tag{8}$$

Therefore, if the number of columns M of $\boldsymbol{\Phi}$ is much smaller than the number of rows N, ie. $M\ll N$, instead of solving for the displacement increment $\Delta\boldsymbol{u}^{(i)}$ from (7) (with a large, sparse matrix), one solves for $\Delta\boldsymbol{x}^{(i)}$ from (8) (with a small, dense matrix) and uses (5) to arrive at $\Delta\boldsymbol{u}^{(i)}$.

Provided the configuration space ($\boldsymbol{u}$) and its tangent space ($\Delta\boldsymbol{u}$) may be identified [2], the modal matrix does not have to possess any relation to the linearization of the equations of motion. This realization is the key to our approach: instead of resorting to the principle of linearized modal motion, we apply a statistical viewpoint of the ***empirical eigenvector basis***.

Empirical Eigenvector Basis

Empirical eigenvectors (EE) are obtained by a procedure which has roots in statistics, and which appears in the literature under a multitude of names as Karhunen-Loève expansion, principal component analysis, empirical orthogonal eigenfunctions, factor analysis, proper orthogonal decomposition, and total least squares (see References [3] and [2]). The singular value decomposition algorithm is the key to the understanding of these methods.

Consider a solid body subjected to some dynamic loads during a time interval I. Mark arbitrary N points on the surface of the body and/or in its interior, and record the positions of these N points at S instants during the interval I. Collect the observations into an ensemble of $3N$-dimensional vectors, $\boldsymbol{u}^j\in\mathbb{R}^{3N}$, $j=1,\ldots,S$ (there are three displacement components for each of the N points). Center the ensemble by subtracting its average,

$$\bar{\boldsymbol{u}}=\langle\boldsymbol{u}^i\rangle=\frac{1}{S}\sum_{i=1}^{S}\boldsymbol{u}^i\ , \tag{9}$$

and collect the centered vectors as columns of the matrix $\boldsymbol{U}$

$$\underset{N\times S}{\boldsymbol{U}}=\left[\boldsymbol{u}^1-\bar{\boldsymbol{u}},\boldsymbol{u}^2-\bar{\boldsymbol{u}},\ldots,\boldsymbol{u}^S-\bar{\boldsymbol{u}}\right]\ . \tag{10}$$

If an approximation to the vector ensemble $\boldsymbol{u}^j$ is sought in the form

$$\widetilde{\boldsymbol{u}} = \boldsymbol{v}_0 + \sum_{i=1}^{M} w_i(\boldsymbol{u} - \bar{\boldsymbol{u}})\boldsymbol{v}^i \tag{11}$$

with $M < N$, the expectation

$$E\left(\|\boldsymbol{u} - (\boldsymbol{v}_0 + \sum_{i=1}^{M} w_i(\boldsymbol{u} - \bar{\boldsymbol{u}})\boldsymbol{v}^i)\|^2 \right) \tag{12}$$

is minimized by $\boldsymbol{v}_0 = \bar{\boldsymbol{u}}$, $\boldsymbol{v}^i = \boldsymbol{\phi}^i$, and $w_i(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{\phi}^i$, where $\boldsymbol{\phi}^i$ are the orthonormal eigenvectors of the covariance matrix

$$\boldsymbol{C}_d = \frac{1}{M}\boldsymbol{U}\boldsymbol{U}^T \tag{13}$$

corresponding to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. In other words,

$$\widetilde{\boldsymbol{u}} = \bar{\boldsymbol{u}} + \sum_{i=1}^{M} \left[(\boldsymbol{u} - \bar{\boldsymbol{u}})^T\boldsymbol{\phi}^i\right]\boldsymbol{\phi}^i \tag{14}$$

is the best M-dimensional linear approximation to $\boldsymbol{u}$ in the quadratic mean. The approximation error is

$$E\left(||\boldsymbol{u} - \widetilde{\boldsymbol{u}}||^2\right) = \lambda_{M+1} + \ldots + \lambda_N \tag{15}$$

If the number of samples S is smaller than $3N$, the EE's may be computed advantageously by the *method of snapshots*. The sample covariance matrix $\boldsymbol{C}_s$ is constructed as

$$\boldsymbol{C}_s = \frac{1}{M}\boldsymbol{U}^T\boldsymbol{U} \ . \tag{16}$$

The non-zero spectra of $\boldsymbol{C}_s$ and $\boldsymbol{C}_d$ are the same [2].

The above recipe is directly applied to finite element (FE) models by collecting displacements at all FE nodes. The EE's of the ensemble covariance matrix are then used in the incremental Ritz algorithm. The EE's are the *optimal* generalized coordinates for a given FE model and given boundary and initial conditions. However, using the EE's computed for a particular FE model for a similar model as a near-optimal basis is feasible when the response of the two full FE models is not drastically different [2].

Example

A shallow spherical cap clamped at the edge is loaded suddenly by a normal force at the apex. The finite element model used in the present study discretizes one quarter of the shell with 768 hexahedral elements (Figure 1).

Solutions with axisymmetric (hence, one-dimensional) reduced models have been reported by Nickell [4] (20 eigenvibration modes, updated every step), and Noor [5] (5 eigenvibration modes + 5 "steady state" modes). These results are compared in Figure 2 with a reference solution based on a refined finite element solution. While there is a noticeable drift in the apparent period and amplitude reduction in the results reported in the above references, the present approach achieves considerably improved accuracy with a smaller number of modes, which are kept unchanged throughout the simulation.

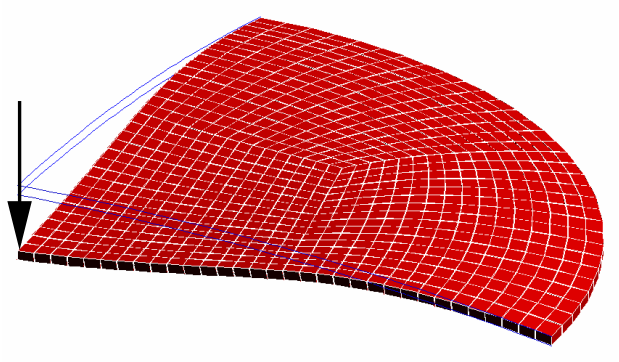

Figure 1: The finite element model of one quarter of a shallow spherical shell (768 hexahedra).

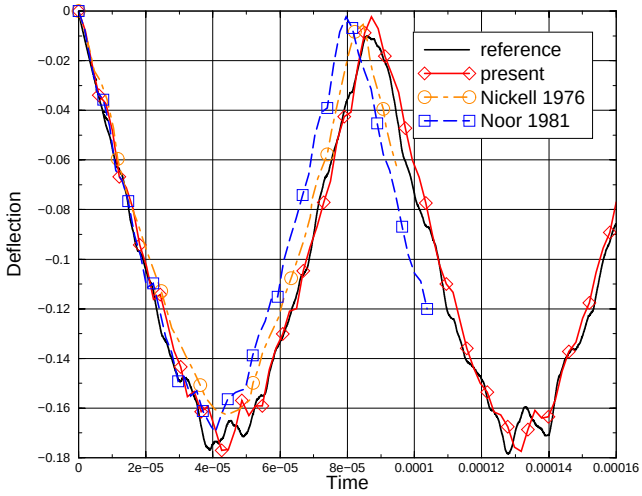

Figure 2: Comparison of results for center deflection. ***Reference***: solution for a refined finite element model; ***Present***: 8-mode reduced model; ***Nickell 1976***: 20 eigenvibration modes, updated every step; ***Noor 1981***: 5 eigenvibration modes + 5 “steady state” modes.

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