Panel 1

Exercise 4-a

List possible combinations of force and displacement boundary conditions for prestressed cable.

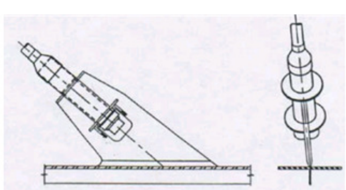
1

Panel 2

Possible combinations of boundary conditions for the prestressed cable

At each end it is possible to prescribe either force or displacement. We have to inspect the particular situation in order to decide which boundary condition applies.

For instance, consider the vibration of the stay cables of this bridge.



Perhaps prescribed displacements would be appropriate.

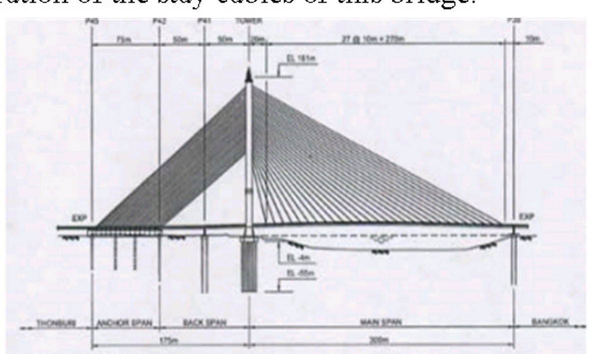


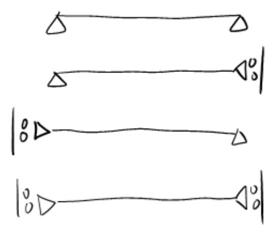
Fig. 2: General Arrangement

2

Panel 3

At each end it is possible to prescribe either force or displacement. We have to inspect the particular situation in order to decide which boundary condition applies.

The possible combinations are:



3

Panel 1

Exercise 4-b

Search the Web of Science for a research article on the equation of motion for prestressed cables that account for the so-called sag. What is the main difference between the model introduced in class and the model you found?

1

Panel 2

Equation of motion that accounts of sag

ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

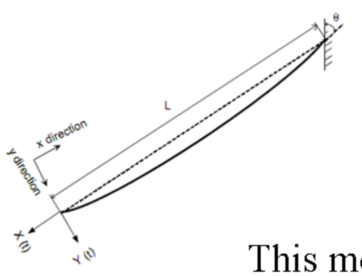
Journal of Sound and Vibration 261 (2003) 403–420

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Response characteristics of local vibrations in stay cables on an existing cable-stayed bridge

Q. Wu, K. Takahashi*, T. Okabayashi, S. Nakamura



$$m\frac{\partial^2 v}{\partial t^2} - P\frac{\partial^2 v}{\partial^2 x} - \Delta P\left(\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v_0}{\partial^2 x}\right) = 0,$$

$$\Delta P = \frac{EA}{L}\left\{u|_{x=L} - u|_{x=0} + \frac{1}{2}\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x + \int_0^L \frac{\partial v}{\partial x}\frac{\partial v_0}{\partial x}\mathrm{d}x\right\},$$

This model is **nonlinear**

2

Panel 1

Exercise 4

Derive the force boundary condition at x=0.

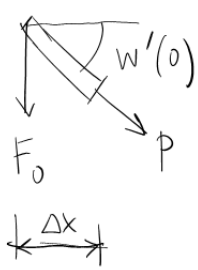
1

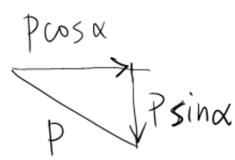
Panel 2

derivation of force boundary condition at x=0

$$\tan\alpha = w'(0)$$

$$\alpha = \mathrm{atan}\, w'(0)$$





The angle α is very small, since the deflections are also very small

$$\alpha << 1 \Rightarrow \cos\alpha \doteq 1$$

$$\sin\alpha \doteq \tan\alpha$$

Vertical equilibrium

$$\downarrow F_0 + \downarrow P\, w'(0) = 0$$

$$P\frac{\partial w(0,t)}{\partial x} + F_0 = 0 \quad \checkmark$$

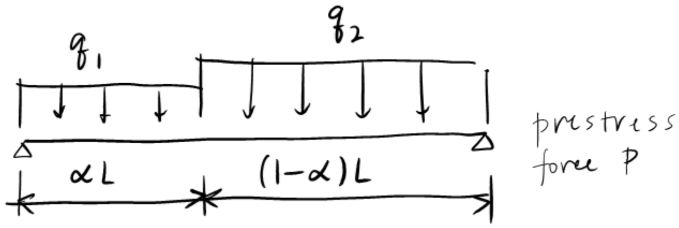
2

Panel 1

Exercise 5-a

Prestressed cable with piecewise uniform load

Solve analytically for the static deflection of the shown prestressed cable.



1

Panel 2

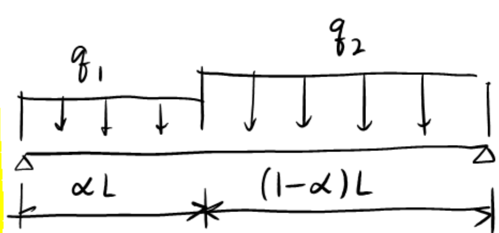
Solving analytically for the static deflection of the shown prestressed cable amounts to solving the following boundary value problem:

$$P w'' + q_1 = 0 \qquad 0 \le x \le \alpha L$$

$$P w'' + q_2 = 0 \qquad \alpha L \le x \le L$$

$$w(0) = 0$$

$$w(L) = 0$$



The second derivative is in general discontinuous where the load changes value. The first derivative however must be continuous at that point, otherwise the second derivative would be infinite (the so-called Dirac Delta Spike), and the equation of motion (equilibrium equation) could not be satisfied at that point.

2

Panel 3

The solution may be derived by writing a general quadratic polynomial within each interval, and then enforcing continuity and the zero deflection at the endpoints. This will provide us with four constants of integration from which four constants of integration maybe determined.

$$0 \le x \le \alpha L$$

$$w(x) = A_1 + B_1 x + q_1 C_1 x^2$$

The quadratic term in the polynomial must be determined so that it satisfies the equilibrium equation. $p w'' + q = 0$

$$w'(x) = B_1 + 2 q_1 C_1 x$$

$$w''(x) = 2 q_1 C_1$$

$$P\, 2 q_1 C_1 + q_1 = 0 \Rightarrow C_1 = -\frac{1}{2P}$$

$$w(x) = A_1 + B_1 x - q_1 \frac{x^2}{2P}$$

3

Panel 4

Similarly

$$\alpha L \le x \le L$$

$$w(x) = A_2 + B x_2 - q_2 \frac{x^2}{2P}$$

Now we introduce the boundary conditions and the continuity conditions. First the boundary conditions:

$$w(0) = 0 \qquad w(L) = 0 \qquad (BC)$$

4

Panel 5

Now the continuity conditions: first the deflection

$$w(\alpha L) = A_1 + B_1 \alpha L - q_1 \frac{(\alpha L)^2}{2P}$$

this is the deflection from the left of αL

$$w(\alpha L) = A_2 + B_2 \alpha L - q_2 \frac{(\alpha L)^2}{2P}$$

this is the deflection from the right of αL

Next the slope:

$$w'(\alpha L) = B_1 - q_1 \frac{\alpha L}{P}$$

this is the slope from the left of αL

$$w'(\alpha L) = B_2 - q_2 \frac{\alpha L}{P}$$

this is the slope from the right of αL

5

Panel 6

Symbolic solution for the integration constants

```
% Solve for the deflection of a prestressed cable
with piecewise uniform
% distributed load
syms A1  B1 A2  B2  alpha  L P q1 q2 x real
w1 = @(x)(A1+B1*x-q1*x^2/(2*P));
dw1 = @(x)(B1-q1*x/P);
w2 = @(x)(A2+B2*x-q2*x^2/(2*P));
dw2 = @(x)(B2-q2*x/P);
```

} symbolic functions

Now solve the system of BC's + continuity eqns

```
Solution =solve([char(w1(0)) '=0'],...
    [char(w2(L)) '=0'],...
    [char(w1(alpha*L)) '=' char(w2(alpha*L))],...
    [char(dw1(alpha*L)) '=' char(dw2(alpha*L))],...
    'A1','B1','A2','B2');
 A1 =Solution.A1;
 B1 =Solution.B1;
 A2 =Solution.A2;
 B2 =Solution.B2;
```

$A_1 = 0 \quad B_1 = \frac{L}{2P}\left(q_2 + 2\alpha q_1 - 2\alpha q_2 - \alpha^2 q_1 + \alpha^2 q_2\right)$

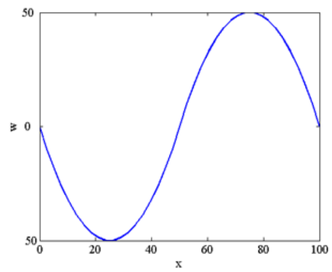
$A_2 = \frac{L^2}{2P}\alpha^2 (q_1 - q_2) \quad B_2 = \frac{L}{2P}\left(q_2 - \alpha^2 q_1 + \alpha^2 q_2\right)$

6

Panel 7

Plot the deflection curve for some particular problem data

```
% Take some particular numbers
alpha = 0.5; q1 =  4; q2 = -4; L = 100; P = 25;
x= linspace(0,alpha*L, 20);
plot (x,-eval((A1+B1*x-q1*x.^2/(2*P))));
hold on
x= linspace(alpha*L,L, 20);
plot (x,-eval((A2+B2*x-q2*x.^2/(2*P))));
Labels =get (gca,'yticklabel');
for i= 1:length(Labels)
    Labels (i,:) =strrep (Labels (i,:),'-',' ');
end
set (gca,'yticklabel', Labels);
xlabel ('x')
ylabel ('w')
```

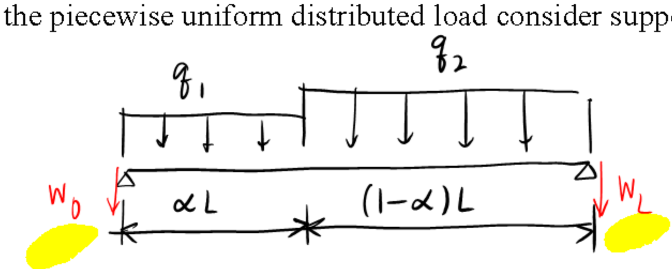


7

Panel 1

Exercise 5-b

Solve analytically for the static deflection of the shown prestressed cable. In addition to the piecewise uniform distributed load consider support settlement.



1

Panel 2

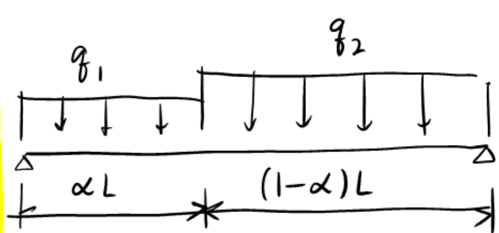
Solving analytically for the static deflection of the shown prestressed cable amounts to solving the following boundary value problem:

$$P w'' + q_1 = 0 \qquad 0 \le x \le \alpha L$$

$$P w'' + q_2 = 0 \qquad \alpha L \le x \le L$$

$$w(0) = w_0$$

$$w(L) = w_L$$



The loading is in general discontinuous where the load changes value. The first derivative however must be continuous of that point, otherwise the second derivative would not be defined, and the equation of motion (equilibrium equation) could not be satisfied at that point.

2

Panel 3

Symbolic solution for the integration constants

```
syms A1  B1  A2  B2  alpha  L P q1 q2 x w0 wL real
w1 = @(x)(A1+B1*x-q1*x^2/(2*P));
dw1 = @(x)(B1-q1*x/P);
w2 = @(x)(A2+B2*x-q2*x^2/(2*P));
dw2 = @(x)(B2-q2*x/P);


Solution =solve([char(w1(0)) '=' char(w0)],...
   [char(w2(L)) '='  char(wL)],...
   [char(w1(alpha*L))  '=' char(w2(alpha*L))],...
   [char(dw1(alpha*L))  '=' char(dw2(alpha*L))],...
   'A1','B1','A2','B2');
 A1  =Solution.A1;
 B1  =Solution.B1;
 A2  =Solution.A2;
 B2  =Solution.B2;
```

$A_1 = w_0$

$B_1 = \frac{L}{2P}\left(q_2 + 2\alpha q_1 - 2\alpha q_2 - \alpha^2 q_1 + \alpha^2 q_2\right) - \frac{w_0 - w_L}{L}$

$A_2 = \frac{L^2}{2P}\alpha^2(q_1 - q_2) + w_0$

$B_2 = \frac{L}{2P}\left(q_2 - \alpha^2 q_1 + \alpha^2 q_2\right) - \frac{w_0 - w_L}{L}$

3

Panel 4

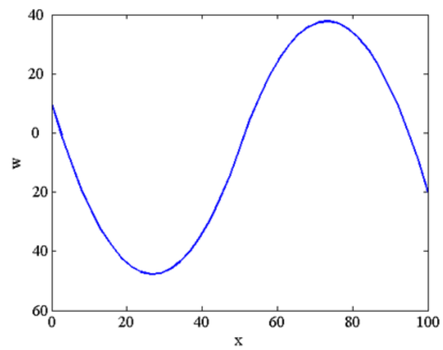
Print out the symbolic solution

```
>> pretty(Solution.A1)
pretty(simplify(Solution.B1))
pretty(simplify(Solution.A2))
pretty(simplify(Solution.B2))

  w0

                                                  2               2
  L (q2 + 2 alpha q1 - 2 alpha q2 - alpha  q1 + alpha  q2)   w0 - wL
  -------------------------------------------------------- - -------
                          2 P                                   L

        2      2
       L  alpha  (q1 - q2)
  w0 + -------------------
               2 P

                2             2
  L (q2 - alpha  q1 + alpha  q2)   w0 - wL
  ------------------------------ - -------
               2 P                    L
>>
```

4

Panel 5

```
% Take some particular numbers
alpha = 0.5; q1 =  4; q2 = -4; L = 100; P = 25;
w0 =-10; wL = 20;
x= linspace(0,alpha*L, 20);
plot (x,-eval((A1+B1*x-q1*x.^2/(2*P))));
hold on
x= linspace(alpha*L,L, 20);
plot (x,-eval((A2+B2*x-q2*x.^2/(2*P))));
Labels =get (gca,'yticklabel');
for i= 1:length(Labels)
    Labels (i,:) =strrep (Labels (i,:),'-',' ');
end
set (gca,'yticklabel', Labels);
xlabel ('x')
ylabel ('w')
```

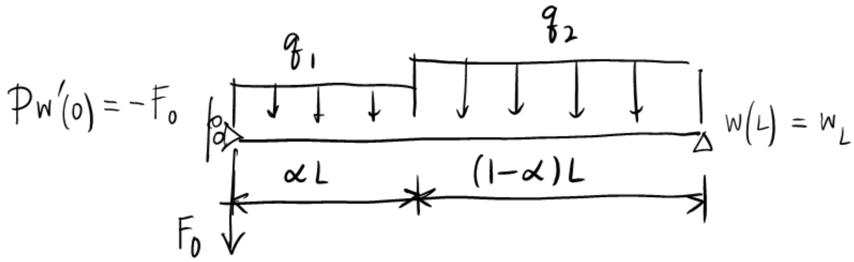


5

Panel 1

Exercise 5-c

Solve analytically for the static deflection of the shown prestressed cable.

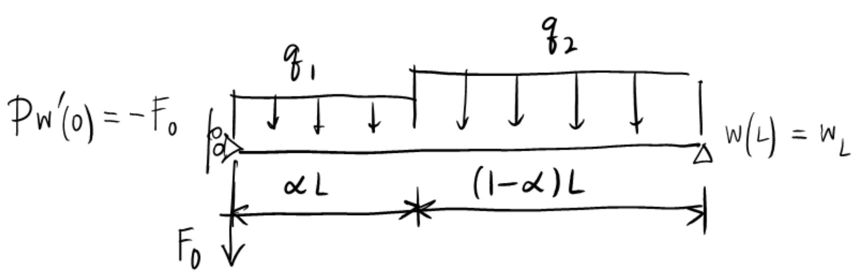


1

Panel 2

Prestressed cable with piecewise uniform load and mixed boundary conditions

Solve analytically for the static deflection of the shown prestressed cable.



The deflections at the ends are in general nonzero. This will affect only the application of the boundary conditions.

$w'(0) = -\frac{F_0}{P}$ $\qquad$ $w(L) = w_L$ $\qquad$ F_0, w_L given

2

Panel 3

```
% Solve for the deflection of a prestressed cable with
piecewise uniform
% distributed load
syms A1  B1 A2  B2  alpha  L P q1 q2 x F0 wL real
w1 = @(x)(A1+B1*x-q1*x^2/(2*P));
dw1 = @(x)(B1-q1*x/P);
w2 = @(x)(A2+B2*x-q2*x^2/(2*P));
dw2 = @(x)(B2-q2*x/P);

Solution =solve([char(dw1(0)) '=' char(-F0/P)],...
    [char(w2(L)) '='  char(wL)],...
    [char(w1(alpha*L))  '=' char(w2(alpha*L))],...
    [char(dw1(alpha*L))  '=' char(dw2(alpha*L))],...
    'A1','B1','A2','B2');
```

note the BC

3

Panel 4

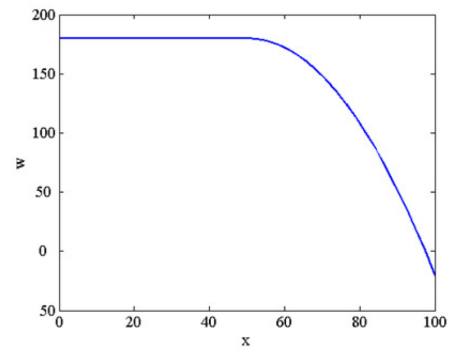
Print out the symbolic solution

```
                 2                2         2           2         2
A1= (2 P wL + L  q2 + 2 F0 L - L  alpha  q1 + L  alpha  q2 +

         2                 2
      2 L  alpha q1 - 2 L  alpha q2) / (2 P)

        F0
B1= -  --
        P

            L (2 F0 + L q2 + 2 L alpha q1 - 2 L alpha q2)
A2=  wL + ---------------------------------------------
                               2 P

       F0 + L alpha q1 - L alpha q2
B2= -  ----------------------------
                    P
>>
```

4

Panel 5

```
% Take some particular numbers
alpha = 0.5; q1 =  0; q2 = -4; L = 100; P = 25;
F0 =0; wL = 20;
```



5

Panel 1

Exercise 14-a

Solve for the approximate deflection of a simply supported prestressed cable with uniform load using the Galerkin method. Take as the trial function basis a single function $N_1(x) = \sin\frac{\pi x}{L}$

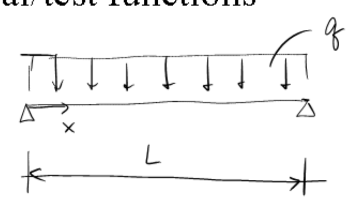
and set the test function $\eta_1 = N_1$.

Compare the midpoint deflection computed analytically and approximately.

1

Panel 2

Solve approximately for the deflection using ad hoc trial/test functions



Balance eqn $Pw'' + q = 0$ (statics)

BC's $w(0) = w(L) = 0$

We limit ourselves to a single basis function. → $N_1(x) = \sin\frac{\pi x}{L}$

Therefore also a single test function. The test function will be chosen the same as the basis function (that is what the Galerkin method does). → $\eta_1(x) = N_1(x)$

The trial function is $w(x) = a_1 N_1(x)$

The coefficient a_1 is the only unknown (which is why we need a single test function -- to derive a single equation from which to solve for unknown).

The trial function must satisfy all essential boundary conditions

The selected trial function does since the basis function does: $N_1(0) = N_1(L) = 0 \Rightarrow w(0) = w(L) = 0$

2

Panel 3

Equation (2.15) simplifies to

$$-\int_0^L \frac{\partial \eta_j}{\partial x} P \sum_{i=1}^{N} \frac{\partial N_i}{\partial x} w_i(\bar{t})\,\mathrm{d}x \quad + \int_0^L \eta_j q\,\mathrm{d}x \quad = 0, \quad j = 1, ..., N$$

In fact $N = 1$

So that we have a single equation

$$-\int_0^L \eta_1' P a_1 N_1' \, dx + \int_0^L \eta_1 q \, dx = 0$$

$$-P a_1 \int_0^L \eta_1' N_1' \, dx + q \int_0^L \eta_1 \, dx = 0$$

3

Panel 4

$$N_1' = \left(\sin\frac{\pi x}{L}\right)' = \frac{\pi}{L}\cos\frac{\pi x}{L}$$

$$\int_0^L \eta_1' N_1' \, dx = \left(\frac{\pi}{L}\right)^2 \int_0^L \cos^2\frac{\pi x}{L} \, dx$$

```
>> int((cos (pi*x/L) )^2,0, L)
ans =
L/2
```

$$\int_0^L \eta_1 \, dx = \int_0^L \sin\frac{\pi x}{L} \, dx =$$

```
>> int((sin(pi*x/L) ),0, L)
ans =
(2*L)/pi
```

$$-P a_1 \int_0^L \eta_1' N_1' \, dx + q \int_0^L \eta_1 \, dx = 0$$

$$-P a_1 \left(\frac{\pi}{L}\right)^2 (L/2) + q\left(\frac{2L}{\pi}\right) = 0 \Rightarrow a_1 = \frac{4}{\pi}\frac{q}{P}\left(\frac{L}{\pi}\right)^2 = \frac{4}{\pi^3}\frac{qL^2}{P}$$

0

4

Panel 5

Compare with analytical solution, $w_{ex} = \frac{q}{2P} x(L-x)$

midpoint deflection

$$w_{ex}\left(\frac{L}{2}\right) = \frac{q}{2P}\frac{L}{2}\left(L-\frac{L}{2}\right) = \frac{q}{2P}\frac{L^2}{4} = \frac{qL^2}{8P} = 0.125\frac{qL^2}{P}$$

$$w\left(\frac{L}{2}\right) = \frac{4}{\pi^3}\frac{qL^2}{P}\sin\left(\frac{\pi\frac{L}{2}}{L}\right) = \frac{4}{\pi^3}\frac{qL^2}{P} \approx 0.129\frac{qL^2}{P}$$

5

Panel 1

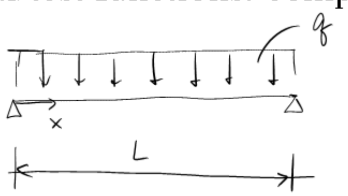
Exercise 14-b

For the approximate deflection computed in exercise 14-a evaluate the balance residual.

1

Panel 2

Solve approximately for the deflection using ad hoc trial/test functions: compute the residual



Balance eqn $P w'' + q = 0$ (statics)

BC's $w(0) = w(L) = 0$

$$N_1(x) = \sin\frac{\pi x}{L}$$

We limit ourselves to a single basis function. Therefore also a single test function. The test function will be chosen the same as the basis function (that is what the Galerkin method does).

$$\eta_1(x) = N_1(x)$$

$$w(x) = a_1 N_1(x) = \frac{4}{\pi^3}\frac{qL^2}{P}\sin\frac{\pi x}{L} \quad \text{(from previous exercise)}$$

$$w'(x) = \frac{4}{\pi^3}\frac{qL^2}{P}\frac{\pi}{L}\cos\frac{\pi x}{L}$$

$$w''(x) = \frac{4}{\pi^3}\frac{qL^2}{P}\left(\frac{\pi}{L}\right)^2\left(-\sin\frac{\pi x}{L}\right) = -\frac{4}{\pi}\frac{q}{P}\sin\frac{\pi x}{L}$$

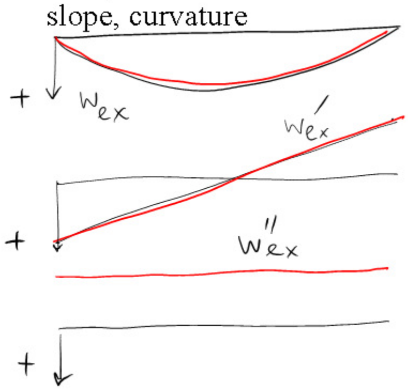
2

Panel 3

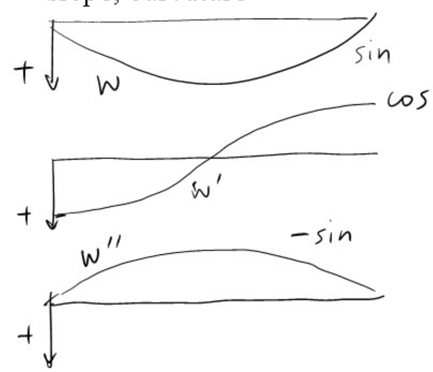
$$P w'' + q = P\left(-\frac{4}{\pi}\frac{q}{P}\sin\frac{\pi x}{L}\right) + q = \left(1 - \frac{4}{\pi}\sin\frac{\pi x}{L}\right) q = r_B$$

$r_B \neq 0$ The trial solution does not satisfy the balance equation

Analytical deflection, slope, curvature



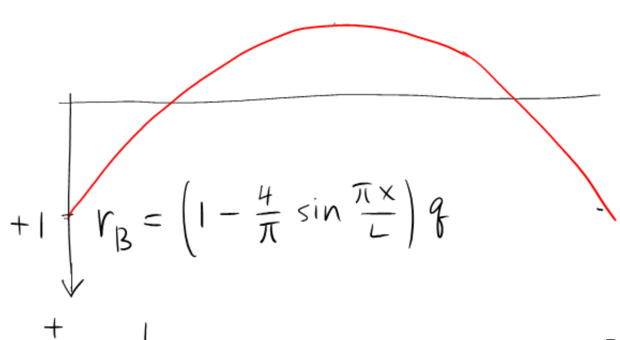
Approximate deflection, slope, curvature



3

Panel 4

The balance equation residual is not zero



$$r_B = \left(1 - \frac{4}{\pi}\sin\frac{\pi x}{L}\right) q$$

Check that $\int_0^L r_B \eta_1 \, dx = 0$

```
syms  L P  x real
% int((cos (pi*x/L) )^2,0, L)
% int((sin(pi*x/L) ),0, L)
rB = 1-(4/pi)*sin(pi*x/L);
int((sin(pi*x/L)*rB ),0, L)
```

↓

0 ✓

4

Panel 1

Exercise 14-c

Solve for the approximate deflection of a simply supported prestressed cable with uniform load using the Galerkin method. Take as the trial function basis the two functions $N_1(x) = \sin\frac{\pi x}{L}$, $N_2(x) = \sin\frac{3\pi x}{L}$

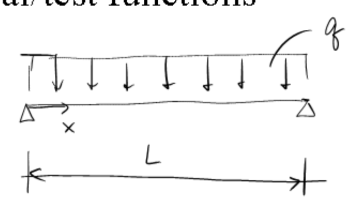
and set the test function $\eta_1 = N_1$, $\eta_2 = N_2$

Compare the midpoint deflection computed analytically and approximately.

1

Panel 2

Solve approximately for the deflection using ad hoc trial/test functions



Balance eqn $Pw'' + q = 0$ (statics)

BC's $w(0) = w(L) = 0$

We limit ourselves to two basis functions. $N_1(x) = \sin\frac{\pi x}{L}$, $N_2(x) = \sin\frac{3\pi x}{L}$

Therefore two test functions will be needed. $\eta_1 = N_1$, $\eta_2 = N_2$

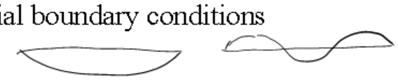
The trial function is

$$w(x) = a_1 N_1(x) + a_2 N_2(x)$$

The coefficients a_1, a_2 are the unknowns .

The trial function must satisfy all essential boundary conditions

The selected trial function does since the basis function does:



2

Panel 3

Equation (2.15) simplifies to

$$-\int_0^L \frac{\partial \eta_j}{\partial x} P \sum_{i=1}^{N} \frac{\partial N_i}{\partial x} w_i(\bar{t})\, \mathrm{d}x \quad + \int_0^L \eta_j q \,\mathrm{d}x \quad = 0, \quad j = 1, \ldots, N$$

In fact $N = 2$

So that we have two equations

$$-\int_0^L \eta_1' P \left(a_1 N_1' + a_2 N_2'\right) dx + \int_0^L \eta_1 q \, dx = 0$$

$$-\int_0^L \eta_2' P \left(a_1 N_1' + a_2 N_2'\right) dx + \int_0^L \eta_2 q \, dx = 0$$

3

Panel 4

These two equations may be rewritten in matrix form using

$$-\underbrace{\int_0^L \eta_1' P N_1' dx}_{K_{11}}\, a_1 - \underbrace{\int_0^L \eta_1' P N_2' dx}_{K_{12}}\, a_2 + \underbrace{\int_0^L \eta_1 q \, dx}_{L_1} = 0$$

$$-\underbrace{\int_0^L \eta_2' P N_1' dx}_{K_{21}}\, a_1 - \underbrace{\int_0^L \eta_2' P N_2' dx}_{K_{22}}\, a_2 + \underbrace{\int_0^L \eta_2 q \, dx}_{L_2} = 0$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

4

Panel 5

$$\int_0^L \eta_1' P N_1' \, dx = P\left(\frac{\pi}{L}\right)^2 \frac{L}{2} = \frac{\pi^2}{2}\frac{P}{L}, \quad \int_0^L \eta_1' P N_2' \, dx = P \frac{\pi}{L}\frac{3\pi}{L} \cdot 0 = 0$$

$$\int_0^L \eta_2' P N_1' \, dx = 0, \quad \int_0^L \eta_2' P N_2' \, dx = P\left(\frac{3\pi}{L}\right)^2 \cdot \frac{L}{2} = \frac{9\pi^2}{2}\frac{P}{L}$$

$$\int_0^L \eta_1 q \, dx = \frac{2L}{\pi} q \qquad \int_0^L \eta_2 q \, dx = \frac{2L}{3\pi} q$$

$$\frac{P}{L}\begin{bmatrix} \frac{\pi^2}{2} & 0 \\ 0 & \frac{9\pi^2}{2} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\pi} \\ \frac{2}{3\pi} \end{bmatrix} qL$$

5

Panel 6

$$\frac{P}{L}\begin{bmatrix} \frac{\pi^2}{2} & 0 \\ 0 & \frac{9\pi^2}{2} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\pi} \\ \frac{2}{3\pi} \end{bmatrix} qL$$

$$a_1 = \frac{qL^2}{P}\frac{2}{\pi} \cdot \frac{2}{\pi^2} = \frac{4}{\pi^3}\frac{qL^2}{P}$$

$$a_2 = \frac{qL^2}{P}\frac{2}{3\pi} \cdot \frac{2}{9\pi^2} = \frac{4}{27\pi^3}\frac{qL^2}{P}$$

So the approximate deflection is

$$w(x) = \frac{4}{\pi^3}\frac{qL^2}{P}\left(\sin\frac{\pi x}{L} + \frac{1}{27}\sin\frac{3\pi x}{L}\right)$$

6

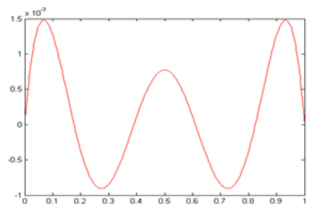
Panel 7

Compare with analytical solution, $w_{ex} = \frac{q}{2P} x(L-x) = \frac{qL^2}{2P} \frac{x}{L}\left(1-\frac{x}{L}\right)$

midpoint deflection

$$w_{ex}\left(\frac{L}{2}\right) = \frac{q}{2P}\frac{L}{2}\left(L-\frac{L}{2}\right) = \frac{q}{2P}\frac{L^2}{4} = \frac{qL^2}{8P} = 0.125\,\frac{qL^2}{P}$$

$$w\left(\frac{L}{2}\right) = \frac{4}{\pi^3}\frac{qL^2}{P}\left(\sin\frac{\pi L/2}{L} + \frac{1}{27}\sin\frac{3\pi L/2}{L}\right) \doteq 0.1242\,\frac{qL^2}{P}$$



This is the difference $(w_{ex} - w)$

7

Panel 1

Exercise 14-d

Solve for the approximate deflection of a prestressed cable with uniform load simply supported at x=0 and with force-free boundary condition at x=L using the Galerkin method. Take as the trial function basis the two functions $N_1(x) = x$, $N_2(x) = x^2$

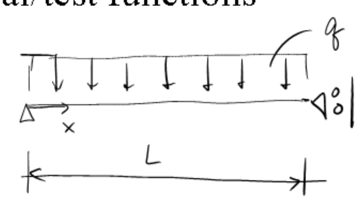
and set the test function $\eta_1 = N_1$, $\eta_2 = N_2$.

Compare the free-end deflection computed analytically and approximately.

1

Panel 2

Solve approximately for the deflection using ad hoc trial/test functions



BVP : $P w'' + q = 0$

$w(0) = 0$

$w'(L) = 0$

We limit ourselves to two basis functions. $N_1(x) = x$, $N_2(x) = x^2$

Therefore two test functions will be needed. $\eta_1 = N_1$, $\eta_2 = N_2$

The trial function is

$$w(x) = a_1 N_1(x) + a_2 N_2(x)$$

The coefficients a_1, a_2 are the unknowns .

The trial function must satisfy all essential boundary conditions
The selected trial function does since the basis function do.

2

Panel 3

Equation (2.15) simplifies to

$$\eta_j(L)F_L - \int_0^L \frac{\partial \eta_j}{\partial x} P \sum_{i=1}^{N} \frac{\partial N_i}{\partial x} w_i(\bar{t})\, \mathrm{d}x + \int_0^L \eta_j q \,\mathrm{d}x = 0, \quad j = 1, \ldots, N \qquad N = 2$$

So that we have two equations $(F_L = 0)$

$$-\int_0^L \eta_1' P \left(a_1 N_1' + a_2 N_2'\right) dx + \int_0^L \eta_1 q \, dx = 0$$

$$-\int_0^L \eta_2' P \left(a_1 N_1' + a_2 N_2'\right) dx + \int_0^L \eta_2 q \, dx = 0$$

$N_1' = 1 \quad , \quad N_2' = 2x$

3

Panel 4

These two equations may be rewritten in matrix form using

$$-\underbrace{\int_0^L \eta_1' P N_1' \, dx}_{K_{11}} \, a_1 - \underbrace{\int_0^L \eta_1' P N_2' \, dx}_{K_{12}} \, a_2 + \underbrace{\int_0^L \eta_1 q \, dx}_{L_1} = 0$$

$$-\underbrace{\int_0^L \eta_2' P N_1' \, dx}_{K_{21}} \, a_1 - \underbrace{\int_0^L \eta_2' P N_2' \, dx}_{K_{22}} \, a_2 + \underbrace{\int_0^L \eta_2 q \, dx}_{L_2} = 0$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

4

Panel 5

$$N_1' = 1 \,,\; N_2' = 2x$$

$$\int_0^L \eta_1' P N_1' \, dx = PL \qquad \int_0^L \eta_1' P N_2' \, dx = P\left[2\frac{x^2}{2}\right]_0^L = PL^2$$

$$\int_0^L \eta_2' P N_1' \, dx = PL^2 \,, \qquad \int_0^L \eta_2' P N_2' \, dx = P\left[4\frac{x^3}{3}\right]_0^L = \frac{4}{3}PL^3$$

$$\int_0^L \eta_1 q \, dx = q\left[\frac{x^2}{2}\right]_0^L = \frac{qL^2}{2} \qquad \int_0^L \eta_2 q \, dx = q\left[\frac{x^3}{3}\right]_0^L = \frac{qL^3}{3}$$

$$PL\begin{bmatrix} 1 & L \\ L & \frac{4}{3}L^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{L}{3} \end{bmatrix} qL^2 \qquad a_1 = \frac{qL}{P} \quad a_2 = -\frac{q}{2P}$$

5

Panel 6

The trial displacement is

$$w(x) = \underbrace{\frac{qL}{P}}_{a_1}\underbrace{x}_{N_1} - \underbrace{\frac{q}{2P}}_{a_2}\underbrace{x^2}_{N_2} = \frac{qL}{P}\left(x - \frac{x^2}{2L}\right)$$

We may suspect that this is actually an exact solution (since it is a second order polynomial). Let us check it. First the balance equation

$$Pw'' = P\left(\frac{qL}{P}\left(x - \frac{x^2}{2L}\right)\right)'' = -q$$

$$Pw'' + q = 0 \;\checkmark$$

Now the boundary conditions.

$$w(0) = 0 \;\checkmark \qquad w'(x) = \frac{qL}{P}\left(1 - \frac{x}{L}\right) \Rightarrow w'(L) = 0 \;\checkmark$$

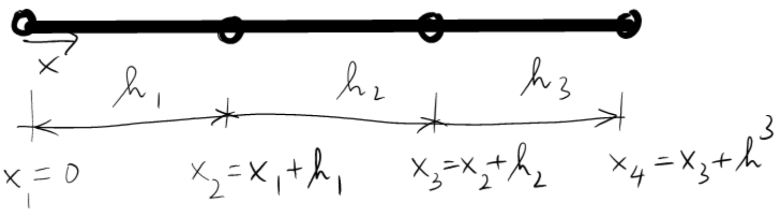
The solution is in fact the exact deflection curve for this boundary value problem.

6

Panel 1

Exercise 15

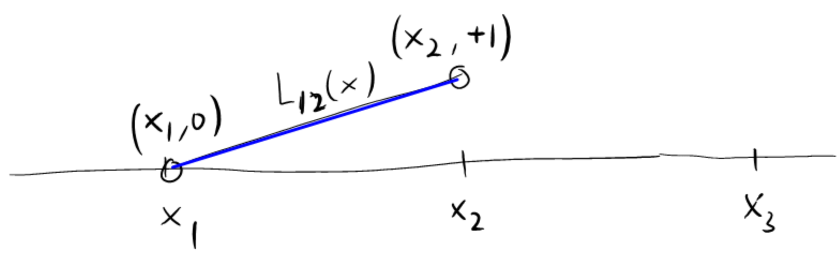
For the shown finite element mesh express the finite element basis functions and their derivatives as expressions in the independent variable x.



1

Panel 2

Lagrange interpolation polynomial



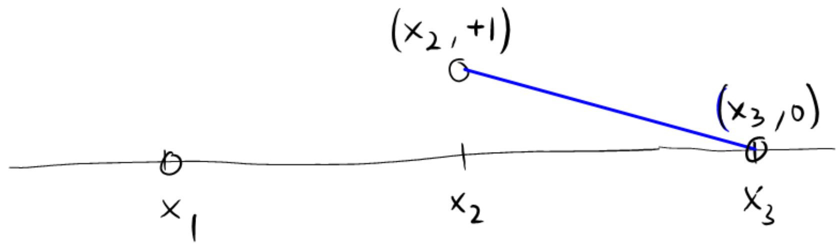
through points $(x_1, 0)$ and $(x_2, +1)$

$$L_{12}(x) = \frac{x - x_1}{x_2 - x_1} \quad \Rightarrow L_{12}(x_1) = \frac{x_1 - x_1}{x_2 - x_1} = 0 \checkmark$$

$$L_{12}(x_2) = \frac{x_2 - x_1}{x_2 - x_1} = 1 \checkmark$$

2

Panel 3



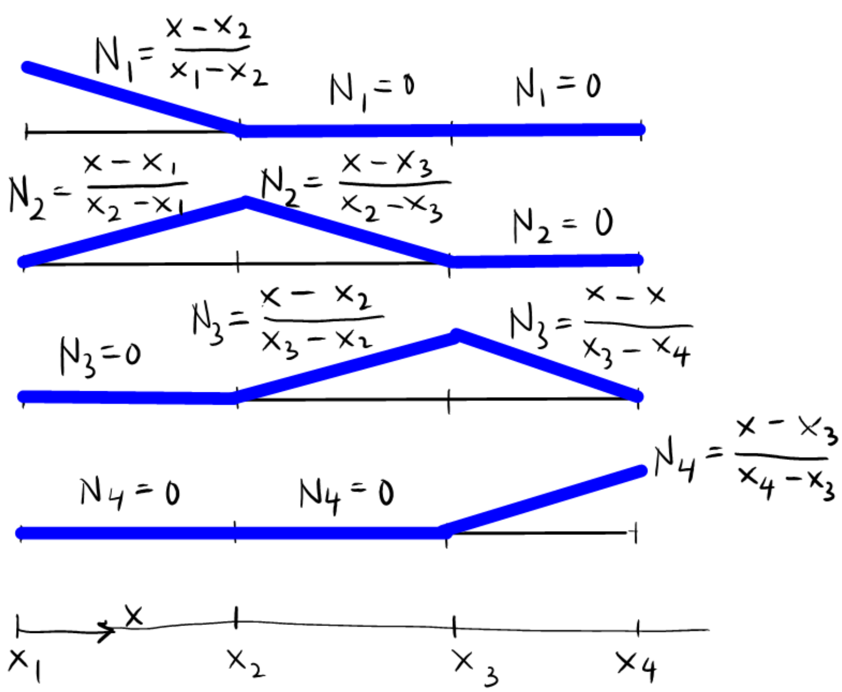
through points $(x_2, +1)$ and $(x_3, 0)$

$$L_{23}(x) = \frac{x - x_3}{x_2 - x_3} \Rightarrow L_{23}(x_2) = \frac{x_2 - x_3}{x_2 - x_3} = 1 \checkmark$$

$$L_{23}(x_3) = \frac{x_3 - x_3}{x_2 - x_3} = 0 \checkmark$$

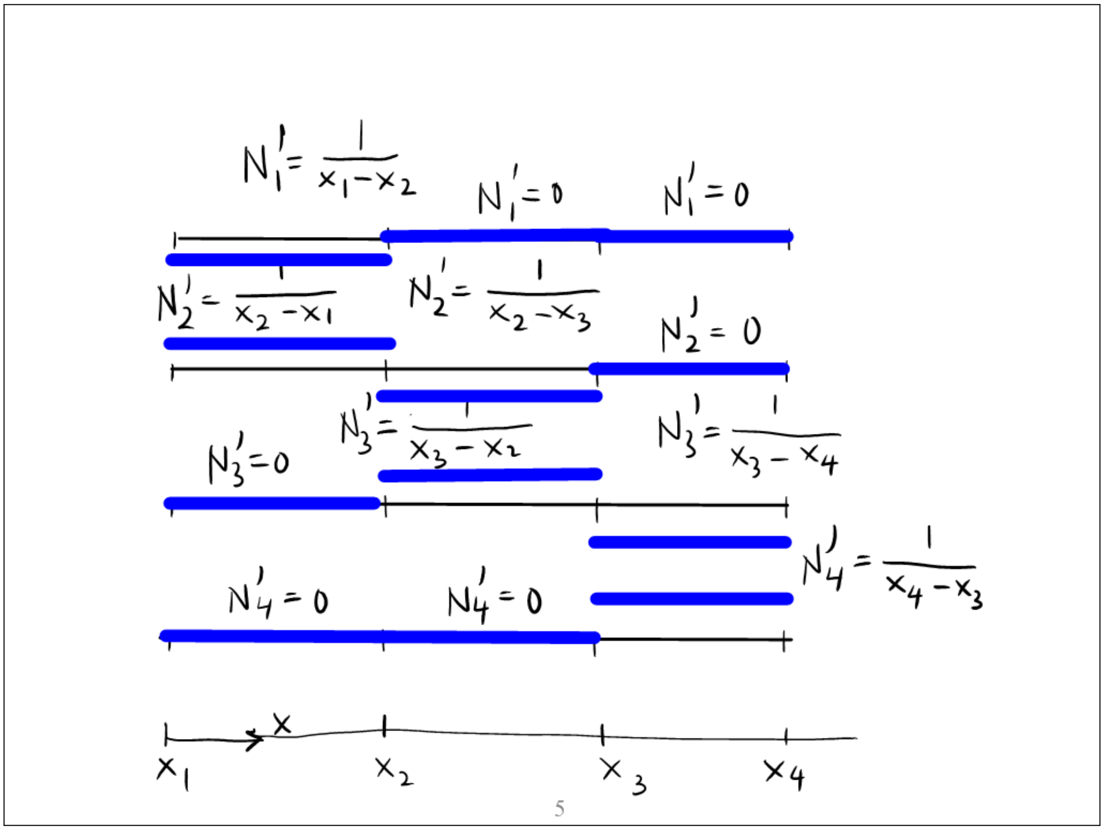
3

Panel 4



4

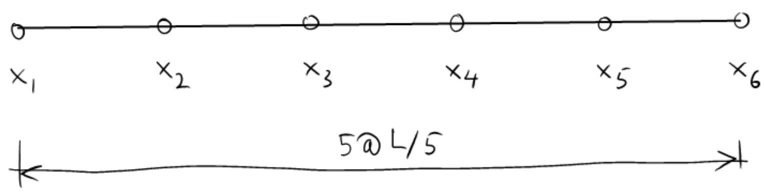
Panel 5


N1' = 1/(x1 − x2)
N1' = 0
N1' = 0
N2' = 1/(x2 − x1)
N2' = 1/(x2 − x3)
N2' = 0
N3' = 0
N3' = 1/(x3 − x2)
N3' = 1/(x3 − x4)
N4' = 0
N4' = 0
N4' = 1/(x4 − x3)
x
x1
x2
x3
x4
5

Panel 1

Exercise 16-a

Interpolate $\cos\left(\frac{2\pi x}{L}\right)$ on the interval $0 \le x \le L$ on a mesh of five equally-sized L2 finite elements.



1

Panel 2

Interpolation is defined by the condition that the interpolating function is equal to the interpolated function at the nodes.

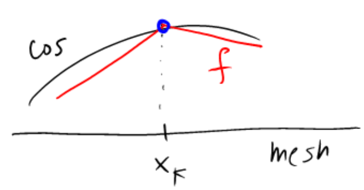
$\cos\left(\frac{2\pi x}{L}\right)$ is the interpolated function

$f(x) = \sum N_i(x) f_i$ is the interpolating function, where f_i are the parameters that we need to determine from the so-called interpolation conditions

The interpolation condition is written as

$$\cos\left(\frac{2\pi x_k}{L}\right) = f(x_k)$$

for all nodes k.



It is very important to realize that the properties of the finite element basis functions make the interpolation very easy. Namely, we have that

$$N_i(x_k) = \begin{cases} +1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

These is the Kronecker delta property

2

Panel 3

Therefore, the value of the interpolating function at the node is

$$f(x_k) = \sum N_i(x_k) f_i$$

Because of the Kronecker property, we have

$$f(x_k) = \sum N_i(x_k) f_i =$$

$$\underbrace{N_1(x_k)}_{0} f_1 + \underbrace{N_2(x_k)}_{0} f_2 + \dots + \underbrace{N_k(x_k)}_{1} f_k + \dots \underbrace{N_{k+1}(x_k)}_{0} f_{k+1} =$$

$$f_k$$

The interpolation condition is recalled as $f(x_k) = \cos\left(\frac{2\pi x_k}{L}\right)$

and therefore the parameters of the finite element interpolation function are $f_k = f(x_k) = \cos\left(\frac{2\pi x_k}{L}\right)$

3

Panel 4

The finite element interpolation function is therefore written on the given mesh as

$$f(x) = \sum_{i=1}^{6} N_i(x) \cos\left(\frac{2\pi x_i}{L}\right)$$

For this mesh we have

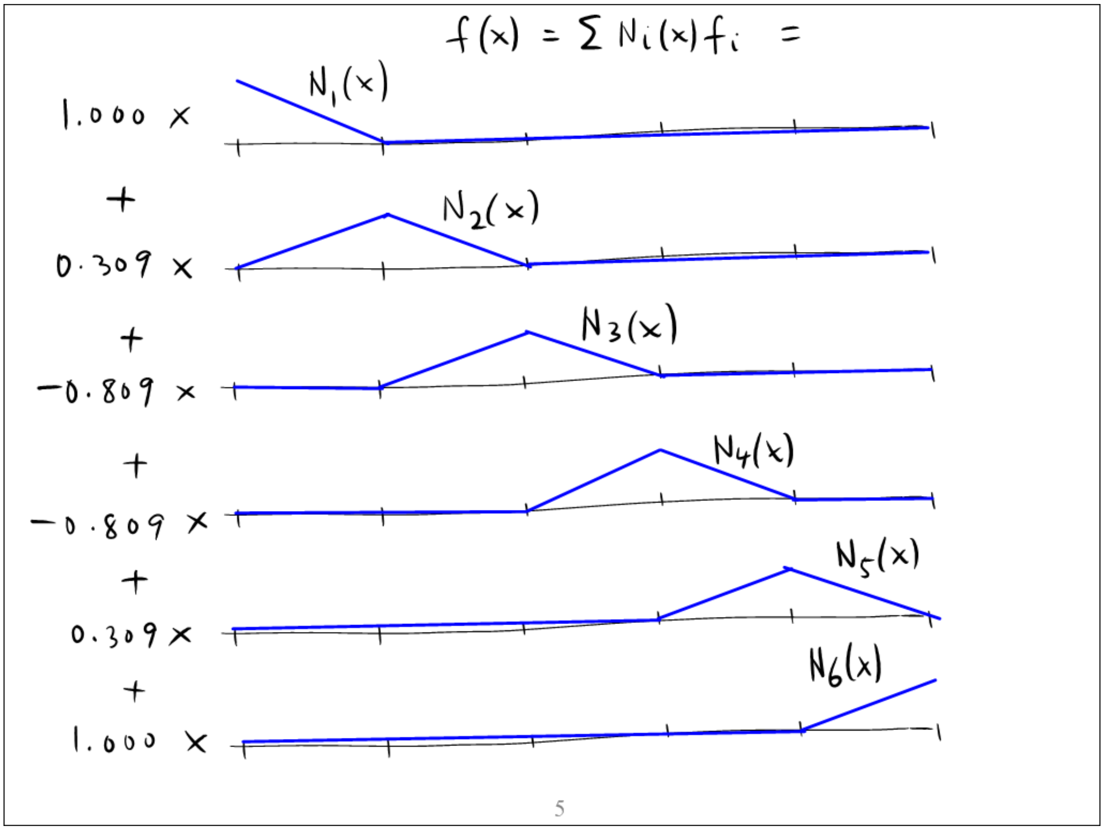
i	1	2	3	4	5	6
$\frac{x_i}{L}$ =	0	0.2000	0.4000	0.6000	0.8000	1.0000

Therefore,

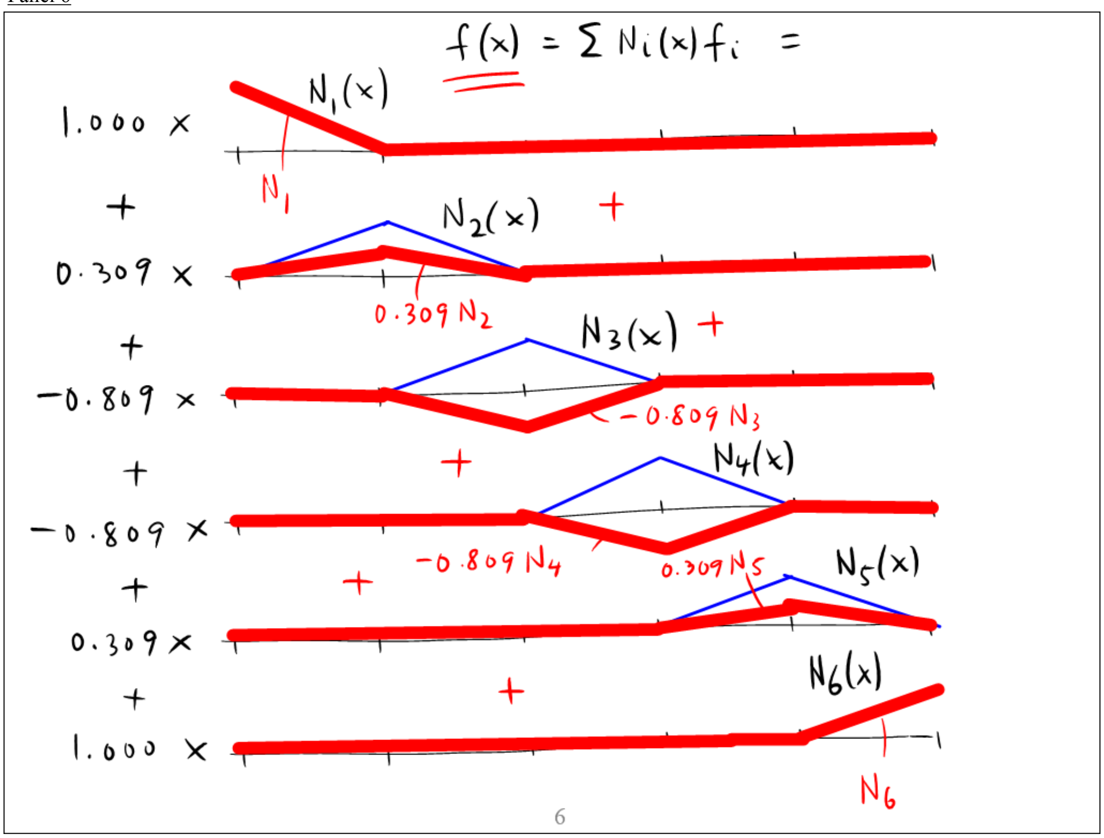
i	1	2	3	4	5	6
$\cos\left(2\pi \frac{x_i}{L}\right)$ =	1.0000	0.3090	-0.8090	-0.8090	0.3090	1.0000

4

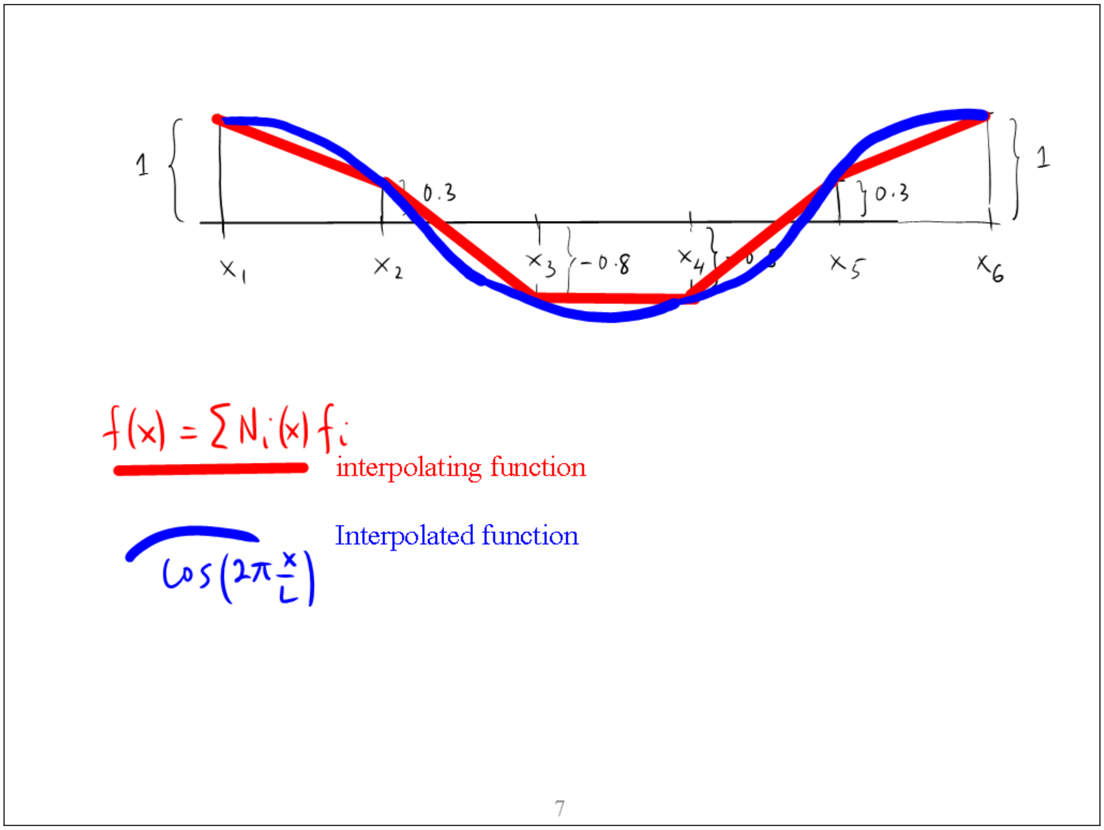
Panel 5



Panel 6



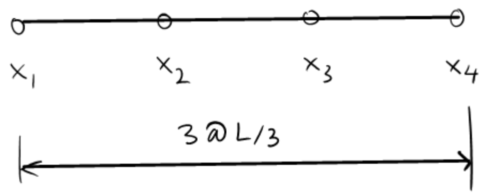
Panel 7


1
0.3
-0.8
0.3
1
x1
x2
x3
x4
x5
x6
f(x) = Σ Ni(x) fi
interpolating function
Interpolated function
cos(2π x/L)
7

Panel 1

Exercise 16-b

Interpolate $ax + b$ on the interval $0 \le x \le L$ on a mesh of three equally-sized L2 finite elements. Show that the interpolation error is zero. In other words, shows that the linear function can be interpolated exactly on the mesh of L2 finite elements.



1

Panel 2

Interpolation is defined by the condition that the interpolating function is equal to the interpolated function at the nodes.

$ax + b$ is the interpolated function

$f(x) = \sum N_i(x) f_i$ is the interpolating function, where f_i are the parameters that we need to determine from the so-called interpolation conditions

The interpolation condition is written as

$$a x_k + b = f(x_k)$$

for all nodes k.

The properties of the finite element basis functions make the interpolation very easy. Namely, we have that

$$N_i(x_k) = \begin{cases} +1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

This is the Kronecker delta property

2

Panel 3

Therefore, the value of the interpolating function at the node is

$$f(x_k) = \sum N_i(x_k)\, f_i$$

Because of the Kronecker property, we have

$$f(x_k) = \sum N_i(x_k) f_i =$$

$$\underbrace{N_1(x_k)}_{0} f_1 + \underbrace{N_2(x_k)}_{0} f_2 + \dots + \underbrace{N_k(x_k)}_{1} f_k + \dots \underbrace{N_{k+1}(x_k)}_{0} f_{k+1} =$$

$$f_k$$

The interpolation condition is recalled as $f(x_k) = a x_k + b$

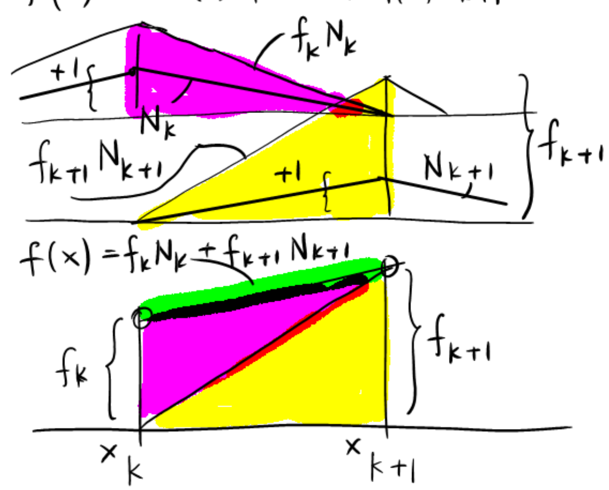
and therefore the parameters of the finite element interpolation function are $f_k = f(x_k) = a x_k + b$

3

Panel 4

Recall that the finite element basis functions are piecewise linear and there are non-zero only in the two finite elements that share a node. Therefore, we have for the interpolating function in the interval $x_k \le x \le x_{k+1}$

$$f(x) = N_k(x) f_k + N_{k+1}(x) f_{k+1}$$



4

Panel 5

Importantly, we see that in the interval $x_k \le x \le x_{k+1}$ the function $f(x)$ is some of two linear functions N_k, N_{k+1}.

Therefore, $f(x)$ is a linear function, and as such it is the unique linear function passing through two given points.

Since the interpolated function $ax + b$ also passes through the same two points, we conclude that the interpolated function is identical to the interpolating function.

Therefore, we conclude that on a mesh of L2 finite elements the basis functions can interpolate exactly arbitrary linear functions.

5

Panel 1

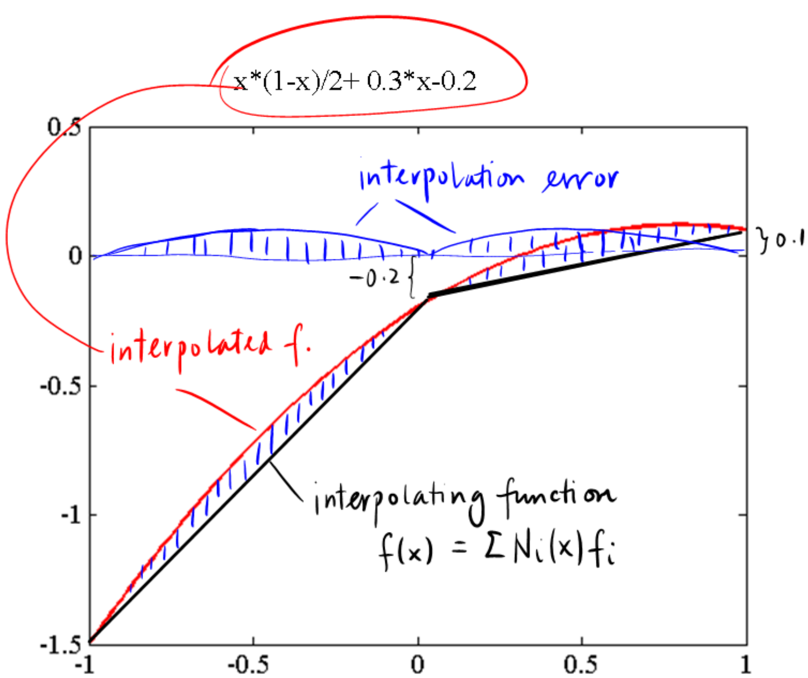
Exercise 16-c

Illustrate the error of interpolation of an arbitrary quadratic function when it is interpolated by a finite element expansion using a mesh of L2 finite elements.

Plot the interpolation error for both the quadratic function itself and for its derivative (slope).

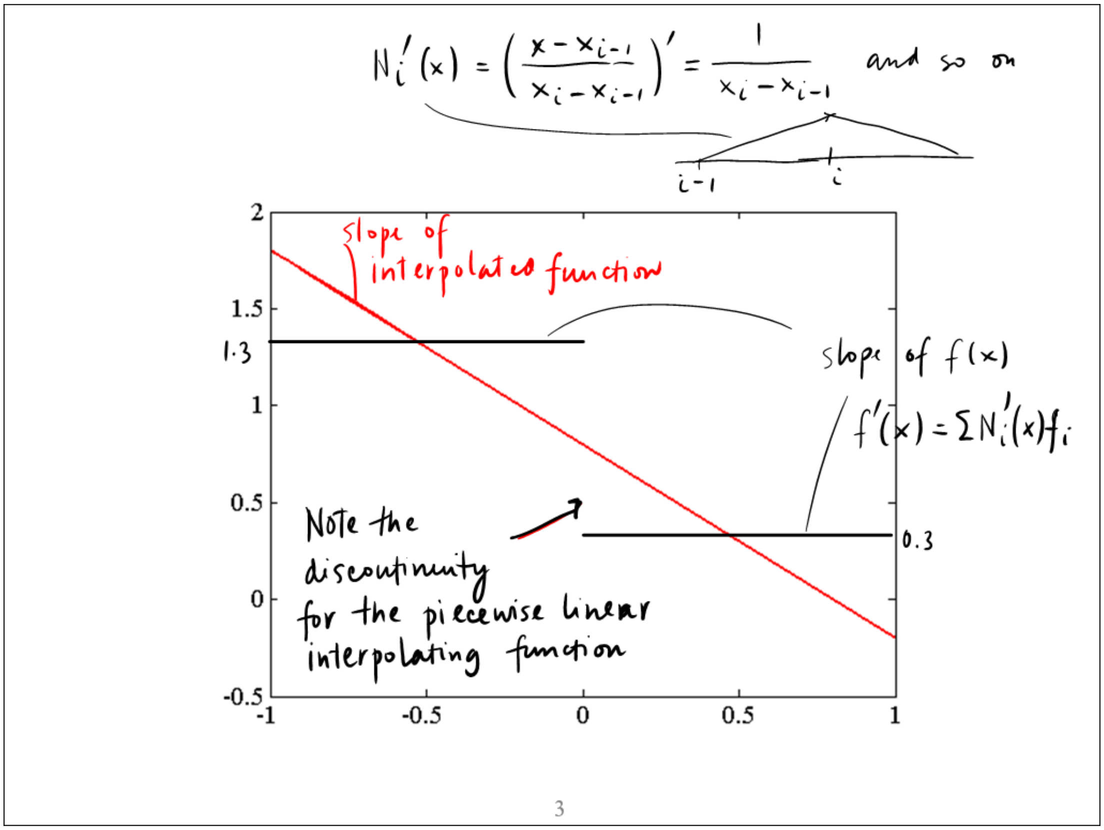
1

Panel 2



2

Panel 3



$N_i'(x) = \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)' = \frac{1}{x_i - x_{i-1}}$ and so on
i-1
i
slope of interpolated function
1.3
slope of f(x)
$f'(x) = \sum N_i'(x) f_i$
0.3
Note the discontinuity for the piecewise linear interpolating function
2
1.5
1
0.5
0
-0.5
-1
-0.5
0
0.5
1
3

Panel 1

Exercise 19-a

Integrate the function $f(x) = 2x^2 + \frac{x^3}{3}$ from -1 to 0 using (1) Trapezoidal rule, (2) Simpson's rule. Compare with the analytical solution.

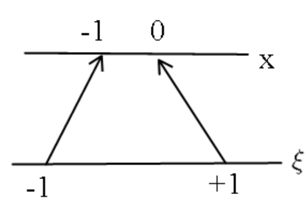
1

Panel 2

The analytical solution is $\int_{-1}^{0} 2x^2 + \frac{x^3}{3} = 7/12 = 0.583333333333333$

The trapezoidal rule has a table on the standard interval $-1 \le \xi \le +1$

ξ_k	W_k
-1	+1
+1	+1



The Jacobian is $\frac{0-(-1)}{2} = \frac{1}{2}$

The two quadrature points map to the ends of the interval. Therefore, with a trapezoidal rule the integral is approximated as

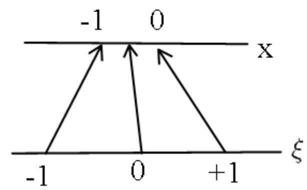
```
f=@(x)2*x^2+(x^3)/3;
 (1/2)*(1*f(-1)+1*f(0))

ans =
   0.833333333333333
```

2

Panel 3

The Simpson's rule has a table on the standard interval $-1 \le \xi \le +1$

ξ_k	W_k
-1	+1/3
0	+4/3
+1	+1/3



Therefore, we can express (refer also to the equation on page 19)

Integrated function

this is (2.24)

```
>> f=@(x)2*x^2+(x^3)/3;
a=-1; b=0;
g=@(xi)(1/2)*(a+b) +(1/2)*(b-a)*xi;
(1/2)*((1/3)*f(g(-1))+(4/3)*f(g(0))+(1/3)*f(g(+1)))

ans =

  0.583333333333333
```

Integral

Jacobian

Note that Simpson's rule gives the exact value of the integral

3

Panel 1

Exercise 19-b

Derive Gaussian quadrature rules using 1, and 2 points on the standard interval.

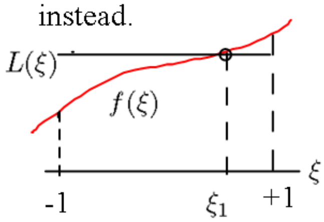
1

Panel 2

We will explain the idea behind the Gaussian quadrature first on the example of a one-point quadrature rule.

The starting point is the idea that the integral of the function $f(\xi)$

may be approximated by integrating its interpolating polynomial $L(\xi)$ instead.



The interpolating polynomial passes through the point ξ_1 and given that the interpolation is through a single point the interpolating polynomial is a constant polynomial.

2

Panel 3

Now we will attempt to make the one-point rule accurate also for a polynomial $p(\xi)$ higher than constant. For instance, we may require that any linear polynomial be integrated exactly by the one-point rule.

In other words, we would be requiring that

$$\sum_{k=1}^{M} L(\xi_k)W_k = L(\xi_1)W_1 = \sum_{k=1}^{M} p(\xi_k)W_k = p(\xi_1)W_1$$

This simply means that the values of the interpolating polynomial $L(\xi)$ and the higher order polynomial $p(\xi)$ must agree at the quadrature point.

$$p(\xi_1) = L(\xi_1)$$

In order for the quadrature rule to give us the exact integral of the function $p(\xi)$ we must then require

$$\int_{-1}^{+1} [p(\xi) - L(\xi)] = 0$$

3

Panel 4

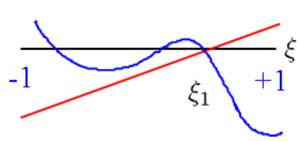
This tells us that we have to find the location of the quadrature point ξ_1 so that the polynomial $p(\xi) - L(\xi)$ passes through zero at the quadrature point, and at the same time it integrates to equal to zero

$$\int_{-1}^{+1} [p(\xi) - L(\xi)] = 0$$

For convenience define

$$F(\xi) = [p(\xi) - L(\xi)]$$

Here are some candidate polynomials that pass through zero at ξ_1



They can be all written as

$$(\xi - \xi_1)q(\xi)$$

where the first term makes sure the product becomes zero at ξ_1

4

Panel 5

So we are trying to satisfy this condition.

$$\int_{-1}^{+1} F(\xi) = \int_{-1}^{+1} (\xi - \xi_1) q(\xi) = 0$$

Consider as an example $q(\xi) = (A\xi + B)$

Then the condition will actually split into two parts each of which needs to be satisfied separately (since the two terms are linearly independent).

$$\int_{-1}^{+1} (\xi - \xi_1)(A\xi + B) = \int_{-1}^{+1} [(\xi - \xi_1) A\xi] + \int_{-1}^{+1} (\xi - \xi_1) B = 0$$

However we cannot satisfy both equations, since we have only one parameter, ξ_1

We can satisfy one condition, which means we can take $q(\xi) = B$

It immediately follows that the solution is $\xi_1 = 0$

5

Panel 6

The weight for the one-point Gaussian rule needs to be determined so that the Lagrange interpolation polynomial itself is integrated exactly. Since here the Lagrange interpolation polynomial is a constant, the weight follows as $W_1 = 2$

It follows from our derivation that the one-point Gaussian rule will be able to integrate exactly linear polynomials.

6

Panel 7

Now let us look at a two-point Gaussian rule. First we will determine the locations of the quadrature points

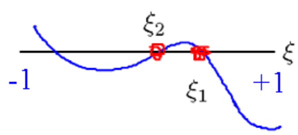
We have to find the location of the quadrature points ξ_1, ξ_2 so that the polynomial $p(\xi) - L(\xi)$ passes through zero at the quadrature point, and at the same time it integrates to equal to zero

$$\int_{-1}^{+1} [p(\xi) - L(\xi)] = 0$$

For convenience define

$$F(\xi) = [p(\xi) - L(\xi)]$$

Here are some candidate polynomials that pass through zero at ξ_1, ξ_2



They can be all written as

$$(\xi - \xi_1)(\xi - \xi_2)q(\xi)$$

where the first two terms makes sure the product becomes zero at ξ_1, ξ_2

7

Panel 8

Now it will be possible to take $q(\xi) = (A\xi + B)$

The integral

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(A\xi + B) = \int_{-1}^{+1} [(\xi - \xi_2)(\xi - \xi_1)A\xi] + \int_{-1}^{+1} (\xi - \xi_2)(\xi - \xi_1)B = 0$$

splits into

$$\int_{-1}^{+1} [(\xi - \xi_2)(\xi - \xi_1)A\xi] = 0, \quad \int_{-1}^{+1} (\xi - \xi_2)(\xi - \xi_1)B = 0$$

which can be solved for the locations of the quadrature points to give

8

Panel 9

```
syms xi xi1 xi2 real
int((xi-xi1)*(xi-xi2),-1,+1),
int(xi*(xi-xi1)*(xi-xi2),-1,+1)
solution =solve('2/3+2*xi1*xi2=0','-2/3*xi1-2/3*xi2=0','xi1','xi2')
solution.xi1
solution.xi2
ans =

 -1/3*3^(1/2)
  1/3*3^(1/2)
```

The locations of the two quadrature points are seen to be $\xi_1 = -\frac{1}{\sqrt{3}}, \xi_2 = \frac{1}{\sqrt{3}}$

9

Panel 10

The weights of the quadrature points are determined so that an arbitrary linear polynomial (the Lagrange interpolation polynomial through two points) is integrated exactly

$$\int_{-1}^{+1} (A\xi + B) = 2B$$

The quadrature formula gives

$$\sum_{k=1}^{M} (A\xi_k + B)W_k = A(-\frac{1}{\sqrt{3}}W_1 + \frac{1}{\sqrt{3}}W_2) + (W_1 + W_2)B$$

Evidently, the exact integral is obtained if $W_1 = 1, W_2 = 1$

10

Panel 11

Number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{1/3}$	1
3	0	$8/9$
	$\pm\sqrt{3/5}$	$5/9$
4	$\pm\sqrt{\left(3-2\sqrt{6/5}\right)/7}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\left(3+2\sqrt{6/5}\right)/7}$	$\frac{18-\sqrt{30}}{36}$
5	0	$128/225$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{10/7}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{10/7}}$	$\frac{322-13\sqrt{70}}{900}$

11

Panel 1

Exercise 19-c

Derive Gaussian 3-point quadrature rule on the standard interval.

1

Panel 2

First we will determine the locations of the quadrature points

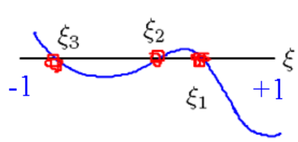
We have to find the location of the quadrature points ξ_1, ξ_2, ξ_3 so that the polynomial $p(\xi) - L(\xi)$ passes through zero at the quadrature point, and at the same time it integrates to equal to zero

$$\int_{-1}^{+1} [p(\xi) - L(\xi)] = 0$$

For convenience define

$$F(\xi) = [p(\xi) - L(\xi)]$$

The candidate polynomials that pass through zero at ξ_1, ξ_2, ξ_3



They can be all written as

$$(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)q(\xi)$$

where the first three terms makes sure the product becomes zero at ξ_1, ξ_2, ξ_3

2

Panel 3

In order to obtain three equations we take $q(\xi) = (A\xi^2 + B\xi + C)$ where A,B,C are arbitrary real numbers.

The integral $\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(A\xi^2 + B\xi + C) = 0$

splits into

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(A\xi^2) = 0$$

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(B\xi) = 0$$

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(C) = 0$$

This constitutes a system of three equations can be solved for the locations of the quadrature points to give

3

Panel 4

```
syms xi xi1 xi2 xi3 real
int((xi-xi1)*(xi-xi2)*(xi-xi3),-1,+1),
int(xi*(xi-xi1)*(xi-xi2)*(xi-xi3),-1,+1)
int(xi^2*(xi-xi1)*(xi-xi2)*(xi-xi3),-1,+1)
solution =solve('-2/3*xi1-2/3*xi2-2/3*xi3-2*xi1*xi2*xi3=0',...
    '2/5+2/3*xi1*xi2-2/3*(-xi1-xi2)*xi3=0',...
    '-2/5*xi1-2/5*xi2-2/5*xi3-2/3*xi1*xi2*xi3=0','xi1','xi2','xi3')
solution.xi1
solution.xi2
solution.xi3

ans =

  1/5*15^(1/2)
 -1/5*15^(1/2)
  1/5*15^(1/2)
 -1/5*15^(1/2)
             0
             0
```

The locations of the quadrature points are seen to be

$$\xi_1 = -1/5 * 15^{1/2}, \xi_2 = 0, \xi_3 = 1/5 * 15^{1/2}$$

4

Panel 5

The weights of the quadrature points are determined so that an arbitrary quadratic polynomial (the Lagrange interpolation polynomial through three points) is integrated exactly

$$\int_{-1}^{+1} L(\xi) = \int_{-1}^{+1} (a\xi^2 + b\xi + c) = (2/3)a + 2c$$

The quadrature formula gives

$$\sum_{k=1}^{3} (a\xi_k^2 + b\xi_k + c)W_k =$$

$$(a\xi_1^2 + b\xi_1 + c)W_1 + (a\xi_2^2 + b\xi_2 + c)W_2 + (a\xi_3^2 + b\xi_3 + c)W_3$$

$$= a(W_1\xi_1^2 + W_2\xi_2^2 + W_3\xi_3^2) + b(W_1\xi_1 + W_2\xi_2 + W_3\xi_3) + c(W_1 + W_2 + W_3)$$

Evidently, the exact integral is obtained if

$$(W_1\xi_1^2 + W_2\xi_2^2 + W_3\xi_3^2) = 2/3, (W_1\xi_1 + W_2\xi_2 + W_3\xi_3) = 0, (W_1 + W_2 + W_3) = 2$$

Hence, we obtain $W_1 = 5/9, W_2 = 8/9, W_3 = 5/9$

5

Panel 6

Since the integral of the difference between $p(\xi) - L(\xi)$

integrates to zero

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(A\xi^2 + B\xi + C) = 0$$

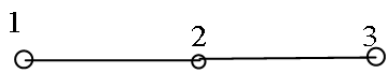
we may conclude that fifth order polynomials (and lower order) will be integrated exactly by the three-point Gaussian rule.

6

Panel 1

Exercise 20-a

Compute the first row of the mass matrix using Gaussian 1-point quadrature for the mesh shown below. Mass density is constant across the mesh.



i	x_i
1	-2
2	1
3	4

Use the Galerkin method (test function = basis function).

1

Panel 2

By definition the mass matrix elements are computed from

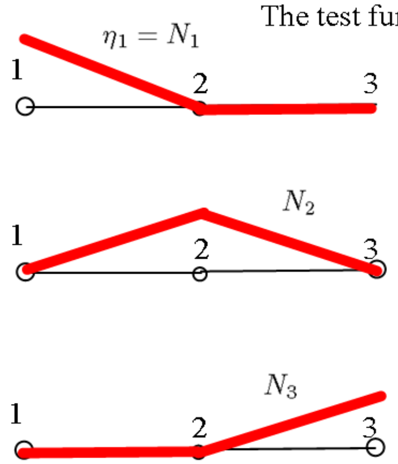
$$M_{ji} = \int_0^L \eta_j \mu N_i \, \mathrm{d}x \ , \qquad (2.17)$$

Because we are using the Galerkin method, $\eta_1 = N_1$

2

Panel 3

The test function/basis function



We can see that functions N_1, N_3 are never both different from zero at any given location. Therefore, the mass matrix element $M_{13} = 0$

3

Panel 4

The mass matrix element M_{11} is evaluated from integration on the first finite element only, since the first basis function is identically zero on the second finite element.

Thus, we need to compute $M_{11} = \int_{x_1}^{x_2} N_1^2 \mu \, dx = \mu \int_{x_1}^{x_2} N_1^2 \, dx$

The basis function may be expressed as the Lagrange interpolation polynomial

$$N_1(x) = \frac{x - x_2}{x_1 - x_2}$$

Analytical integration yields $M_{11} = \mu \int_{x_1}^{x_2} N_1^2 \, dx = \mu(x_2 - x_1)/3$

4

Panel 5

Gaussian one-point integration is according to (2.26) written as

$$M_{11} = \mu N_1(\xi = 0)^2 \times (x_2 - x_1)/2 \times 2$$

Integrand Jacobian Weight

Note that the basis function at the midpoint of the interval assumes the value of one half

$$N_1(\xi = 0) = 1/2$$

We have for the mass matrix element computed with Gaussian one-point quadrature

$$M_{11} = \mu N_1(\xi = 0)^2 \times (x_2 - x_1)/2 \times 2 = \mu(x_2 - x_1)/4$$

5

Panel 6

The mass matrix element M_{12} is evaluated from integration on the first finite element only, since the first basis function is identically zero on the second finite element.

Thus, we need to compute $M_{12} = \int_{x_1}^{x_2} N_1 N_2 \mu \, dx = \mu \int_{x_1}^{x_2} N_1 N_2 \, dx$

The basis functions may be expressed on the first finite element as the Lagrange interpolation polynomials

$$N_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Analytical integration yields $M_{12} = \mu \int_{x_1}^{x_2} N_1 N_2 \, dx = \mu(x_2 - x_1)/6$

6

Panel 7

Gaussian one-point integration is according to (2.26) written as

$$M_{12} = \mu N_1(\xi = 0) N_2(\xi = 0) \times (x_2 - x_1)/2 \times 2$$

Integrand Jacobian Weight

Note that the basis function at the midpoint of the interval assumes the value of one half

$$N_1(\xi = 0) = 1/2 \qquad N_2(\xi = 0) = 1/2$$

We have for the mass matrix element computed with Gaussian one-point quadrature

$$M_{12} = \mu N_1(\xi = 0) N_2(\xi = 0) \times (x_2 - x_1)/2 \times 2 = \mu(x_2 - x_1)/4$$

7

Panel 8

Comparison of the first row of the mass matrix

	M_{11}	M_{12}	M_{13}
Analytical:	$\mu(x_2 - x_1)/3$	$\mu(x_2 - x_1)/6$	0

Numerical (one-point Gaussian quadrature):

$\mu(x_2 - x_1)/4$	$\mu(x_2 - x_1)/4$	0

8

Panel 1

Exercise 20-b

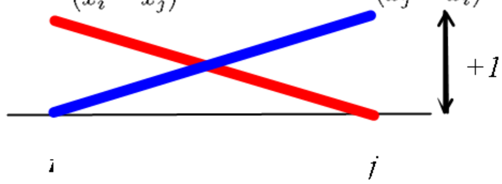
Verify formula (2.28) for the derivatives of the basis functions.

1

Panel 2

The basis functions on the finite element ij may be written in terms of the physical coordinate x using the Lagrange interpolation polynomials as

$$N_i(x) = \frac{(x - x_j)}{(x_i - x_j)} \qquad N_j(x) = \frac{(x - x_i)}{(x_j - x_i)}$$



Consequently, the derivatives of the basis functions on this element may be written as

$$N_i' = \frac{1}{(x_i - x_j)} \qquad N_j' = \frac{1}{(x_j - x_i)}$$

The same results would be obtained from the geometrical picture: the slope of either straight line is rise over run. Rise is either -1 or +1, run is the length of the element, $(x_j - x_i)$

2

Panel 3

In parametric coordinates on the standard interval $-1 \le \xi \le +1$

the basis functions are expressed as shown in equation (2.27). Briefly, the basis functions on the standard interval are Lagrange interpolation functions in terms of the $-1 \le \xi \le +1$ variable.

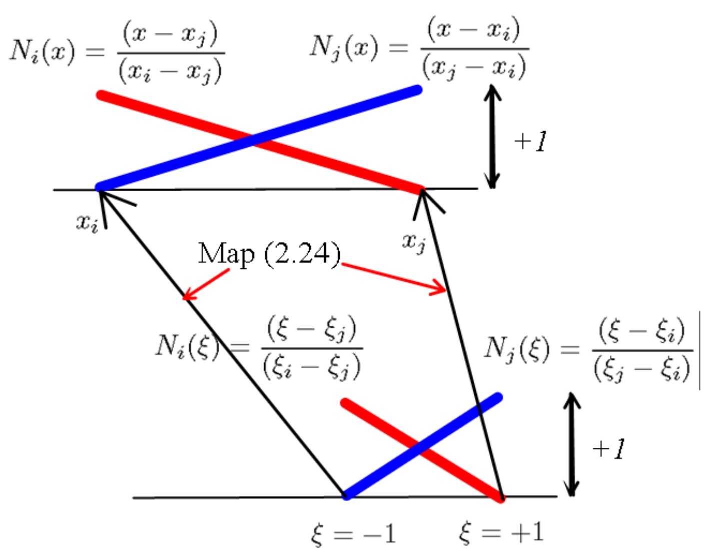
$$N_i(\xi) = \frac{(\xi - \xi_j)}{(\xi_i - \xi_j)} \qquad N_j(\xi) = \frac{(\xi - \xi_i)}{(\xi_j - \xi_i)}$$

Here we write $\xi_i = -1$ for the left-hand side of the standard interval which maps to x_i and $\xi_j = +1$ for the right hand side of the standard interval which maps to x_j

3

Panel 4

We have this picture



4

Panel 5

The derivatives of the basis functions in the parametric coordinates

$$N_i(\xi) = \frac{(\xi - \xi_j)}{(\xi_i - \xi_j)} \qquad N_j(\xi) = \frac{(\xi - \xi_i)}{(\xi_j - \xi_i)}$$

with respect to ξ are readily calculated as

$$\frac{\partial N_i}{\partial \xi} = \frac{1}{(\xi_i - \xi_j)} = -\frac{1}{2} \qquad \frac{\partial N_j}{\partial \xi} = \frac{1}{(\xi_j - \xi_i)} = \frac{1}{2}$$

The map (2.24) is easily inverted: $\xi = \dfrac{2x - a - b}{b - a}$ $\quad a = x_i, b = x_j$

Therefore, the derivative $\dfrac{\partial \xi}{\partial x} = \dfrac{2}{x_j - x_i}$ follows, and we can write

for the derivatives of the basis functions with respect to x

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} = -\frac{1}{2}\frac{2}{x_j - x_i} = -\frac{1}{x_j - x_i} \qquad \frac{\partial N_j}{\partial x} = \frac{\partial N_j}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{2}\frac{2}{x_j - x_i} = \frac{1}{x_j - x_i}$$

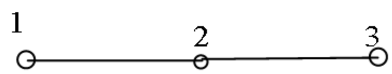
These are the same expressions we obtained previously by directly differentiating the Lagrange interpolation polynomials with the respect to x.

5

Panel 1

Exercise 20-c

Compute the first row of the mass matrix using Gaussian 2-point quadrature for the mesh shown below. Mass density is constant across the mesh.



i	x_i
1	-2
2	1
3	4

Use the Galerkin method (test function = basis function).

1

Panel 2

By definition the mass matrix elements are computed from

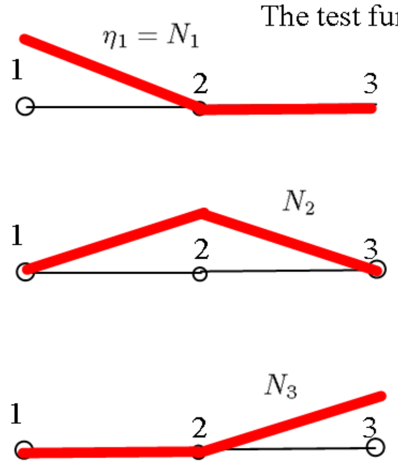
$$M_{ji} = \int_0^L \eta_j \mu N_i \,\mathrm{d}x \;, \qquad (2.17)$$

Because we are using the Galerkin method, $\eta_1 = N_1$

2

Panel 3

The test function/basis function



We can see that functions N_1, N_3 are never both different from zero at any given location. Therefore, the mass matrix element $M_{13} = 0$

3

Panel 4

The mass matrix element M_{11} is evaluated from integration on the first finite element only, since the first basis function is identically zero on the second finite element.

Thus, we need to compute $M_{11} = \int_{x_1}^{x_2} N_1^2 \mu \, dx = \mu \int_{x_1}^{x_2} N_1^2 \, dx$

The basis function may be expressed as the Lagrange interpolation polynomial

$$N_1(x) = \frac{x - x_2}{x_1 - x_2}$$

Analytical integration yields $M_{11} = \mu \int_{x_1}^{x_2} N_1^2 \, dx = \mu(x_2 - x_1)/3$

4

Panel 5

Gaussian 2-point integration is according to (2.26) written as

$$M_{11} = \mu N_1(\xi = 1/\sqrt{3})^2 \times (x_2 - x_1)/2 \times 1$$
$$+\mu N_1(\xi = -1/\sqrt{3})^2 \times (x_2 - x_1)/2 \times 1$$

Integrand Jacobian Weight

The basis function assumes at the quadrature points values of

$$N_1(\xi = -1/\sqrt{3}) = 0.788675134594813 \qquad N_1(\xi = 1/\sqrt{3}) = 0.211324865405187$$

We have for the mass matrix element computed with Gaussian 2-point quadrature

$$M_{11} = \mu(0.211324865405187^2 + 0.788675134594813^2) \times (x_2 - x_1)/2 \times 1 = \mu(x_2 - x_1)/3$$

We can see that the numerical result agrees with the analytical integration.

5

Panel 6

The mass matrix element M_{12} is evaluated from integration on the first finite element only, since the first basis function is identically zero on the second finite element.

Thus, we need to compute $M_{12} = \int_{x_1}^{x_2} N_1 N_2 \mu \, dx = \mu \int_{x_1}^{x_2} N_1 N_2 \, dx$

The basis functions may be expressed on the first finite element as the Lagrange interpolation polynomials

$$N_1(x) = \frac{x - x_2}{x_1 - x_2} \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Analytical integration yields $M_{12} = \mu \int_{x_1}^{x_2} N_1 N_2 \, dx = \mu(x_2 - x_1)/6$

6

Panel 7

Gaussian 2-point integration is according to (2.26) written as

$$M_{12} = \mu N_1(\xi = 1/\sqrt{3}) N_2(\xi = 1/\sqrt{3}) \times (x_2 - x_1)/2 \times 1$$
$$+ \mu N_1(\xi = -1/\sqrt{3}) N_2(\xi = -1/\sqrt{3}) \times (x_2 - x_1)/2 \times 1$$

Integrand Jacobian Weight

The basis function assumes at the quadrature points values of

$N_1(\xi = -1/\sqrt{3}) = 0.788675134594813$ $N_1(\xi = 1/\sqrt{3}) = 0.211324865405187$

$N_2(\xi = -1/\sqrt{3}) = 0.211324865405187$ $N_2(\xi = 1/\sqrt{3}) = 0.788675134594813$

We have for the mass matrix element computed with Gaussian 2-point quadrature

$$M_{12} = \mu(2 * 0.211324865405187 * 0.788675134594813) \times (x_2 - x_1)/2 \times 1 = \mu(x_2 - x_1)/6$$

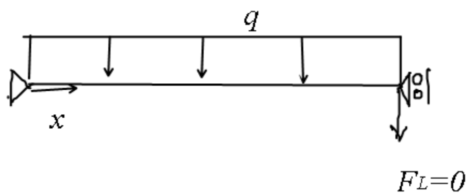
Again, the two-point Gaussian quadrature integrates the mass matrix element exactly.

7

Panel 1

Exercise 28-a

Compute the solution to the problem described in section 3.2 by hand.



1

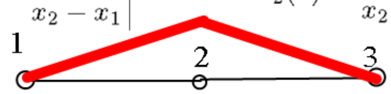
Panel 2

The basis functions (= test functions)

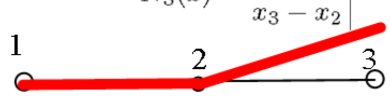
The locations of the nodes

i	*x_i*
1	*0*
2	*L/2*
3	*L*

$$N_2(x) = \left.\frac{x - x_1}{x_2 - x_1}\right| \qquad N_2(x) = \left.\frac{x - x_3}{x_2 - x_3}\right|$$



$$N_3(x) = \left.\frac{x - x_2}{x_3 - x_2}\right|$$



By convention we draw the function above the x-axis when it is positive.

2

Panel 3

The first task is to construct the equations to be solved for the unknown displacements. This means computing the elements of the stiffness matrix and the elements of the load vector.

$$N_j(L)F_L - \sum_{i=1}^{N} K_{ji}w_i + \int_0^L N_j q \,\mathrm{d}x = 0, \quad j = 2, ..., N \; , \tag{3.1}$$

which may be arranged in matrix form as

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{L} \; , \tag{3.2}$$

where $\boldsymbol{K}$ is a square $(N-1)\times(N-1)$ matrix collecting K_{ji}, $i, j = 2, ..., N$. The column matrix $\boldsymbol{d}$ collects the degrees of freedom $d_k = w_{k+1}$, $k = 1, ..., N-1$. The column matrix $\boldsymbol{L}$ is the load vector, with components

$$L_k = N_{k+1}(L)F_L - K_{k+1,1}w_1 + \int_0^L N_{k+1} q \,\mathrm{d}x = 0, \quad k = 1, ..., N-1 \; . \tag{3.3}$$

3

Panel 4

First the load vector. Note that $F_L = 0$ $w_1 = 0$

$1, ..., N-1$. The column matrix $\boldsymbol{L}$ is the load vector, with components

$$L_k = N_{k+1}(L)F_L - K_{k+1,1}w_1 + \int_0^L N_{k+1} q \,\mathrm{d}x = 0, \quad k = 1, ..., N-1 \; . \tag{3.3}$$

Note that the use of the Simpson's rule means that all the integrals will be evaluated exactly since they are at most linear polynomials. Therefore we can evaluate the integrals here analytically, and the result will be identical to that computed with a finite element program in section 3.2.

So for the load vector we obtain

$$L_1 = \int_0^L qN_2(x)\, dx = q(x_3 - x_1)/2 = qL/2$$

$$L_2 = \int_0^L qN_3(x)\, dx = q(x_3 - x_2)/2 = qL/4$$

4

Panel 5

For the stiffness matrix coefficients we have

$$K_{ji} = \int_0^L \frac{\partial \eta_j}{\partial x} P \frac{\partial N_i}{\partial x} \, \mathrm{d}x \; , \qquad (2.16)$$

$$K_{22} = \int_0^L N_2'(x) P N_2'(x) \, dx =$$

$$\int_0^{L/2} \frac{1}{x_2 - x_1} P \frac{1}{x_2 - x_1} \, dx + \int_{L/2}^{L} \frac{1}{x_2 - x_3} P \frac{1}{x_2 - x_3} \, dx =$$

Note that the integral splits into two integrals over each element since the expression for the basis function is different from element to element.

$$\int_0^{L/2} \frac{1}{L/2} P \frac{1}{L/2} \, dx + \int_{L/2}^{L} \frac{1}{L/2} P \frac{1}{L/2} \, dx = \frac{2P}{L} + \frac{2P}{L} = \frac{4P}{L}$$

5

Panel 6

$$K_{23} = \int_0^L N_2'(x) P N_3'(x) \, dx =$$

$$\int_0^{L/2} 0 P \frac{1}{x_3 - x_2} \, dx + \int_{L/2}^{L} \frac{1}{x_2 - x_3} P \frac{1}{x_3 - x_2} \, dx =$$

$$\int_{L/2}^{L} \frac{-1}{L/2} P \frac{1}{L/2} \, dx = \frac{-2P}{L}$$

Note that we have the symmetry $K_{23} = K_{32}$

$$K_{33} = \int_0^L N_3'(x) P N_3'(x) \, dx =$$

$$\int_0^{L/2} 0 P 0 \, dx + \int_{L/2}^{L} \frac{1}{x_3 - x_2} P \frac{1}{x_3 - x_2} \, dx = \int_{L/2}^{L} \frac{1}{L/2} P \frac{1}{L/2} \, dx = \frac{2P}{L}$$

6

Panel 7

Now we can write down the matrix equations

$$\frac{P}{L}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} qL/2 \\ qL/4 \end{pmatrix}$$

with the solution

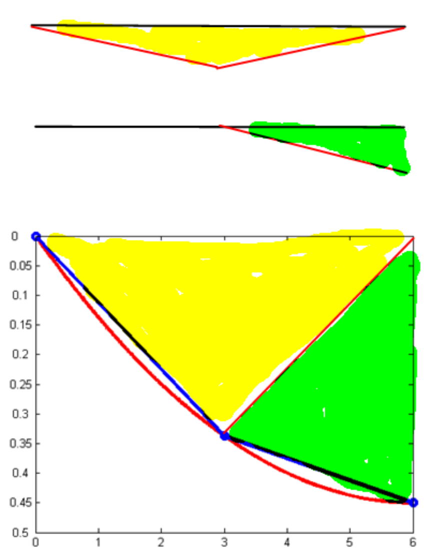
$$\begin{pmatrix} w_2 \\ w_3 \end{pmatrix} = \frac{qL^2}{P}\begin{pmatrix} 3/8 \\ 1/2 \end{pmatrix}$$

7

Panel 8

The solution is displayed on the mesh to mimic the shape of the cable. The deflection function is constructed as

$$w(x) = N_2(x)w_2 + N_3(x)w_3$$

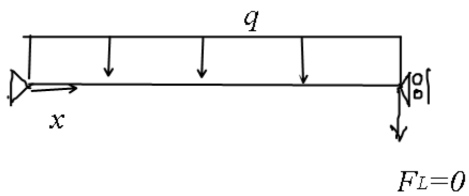


8

Panel 1

Exercise 28-b

Compute the solution to the problem described in section 3.2 by hand. Use the numbering of equations and element-by-element assembly technique.



1

Panel 2

First we will introduce an organizing principle into the definition of the mesh. All elements will be defined by the pair
(left-hand side node, right hand side node)

Element	Nodes
1	1,2
2	2,3

Second, we will assign a numbering to the nodal displacements. We start with the displacements that are unknown (sometimes we say free), and then we follow with the displacements that are known (sometimes we say prescribed).

Node	Unknown #	
1	3	
2	1	$N_1(x)$
3	2	$N_2(x)$

The numbers of the unknowns will determine the numbering of the basis functions.

There are two actual unknowns, the deflection at the left-hand side is known to be zero.

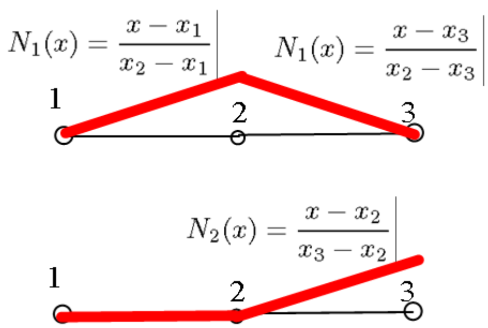
2

Panel 3

The basis functions (= test functions)

The locations of the nodes

i	*x_i*
1	*0*
2	*L/2*
3	*L*



By convention we draw the function above the x-axis when it is positive.

3

Panel 4

Now the load vector components will be computed element-by-element.

$$L_1 = \int_0^L qN_1(x)\,dx = \int_{x_1}^{x_2} qN_1(x)\,dx + \int_{x_2}^{x_3} qN_1(x)\,dx$$

The contribution from these integrals is going to be nonzero only if the nodes of the element are associated with the unknown 1; otherwise the contribution is zero.

This suggests that rather than computing the load vector elements as

Loop over load vector components *j*
    Loop over elements *e*
        Add contribution to component *j* from element *e*

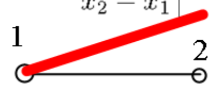
4

Panel 5

we could switch the loops and compute the components of the load vector which are associated with the nodes of the element by looping

Loop over elements *e*
- Add contribution to component *i* associated with first node from element *e*
- Add contribution to component *j* associated with second node from element *e*

Element 1: $N_1(x) = \dfrac{x - x_1}{x_2 - x_1}$



Only one test function is nonzero over this element. LHS node is associated with prescribed displacement.

The contribution to L_1

$$\int_{x_L}^{x_R} qN_1(x)\,dx \qquad x_L = x_1, x_R = x_2$$

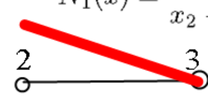
5

Panel 6

$$\int_{x_L}^{x_R} qN_1(x)\,dx = q(x_R - x_L)/2 = q(x_2 - x_1)/2 = qL/4$$

Element 2: $x_L = x_2, x_R = x_3$

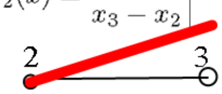
$N_1(x) = \dfrac{x - x_3}{x_2 - x_3}$



The contribution to L_1

$$\int_{x_L}^{x_R} qN_1(x)\,dx = q(x_R - x_L)/2 = q(x_3 - x_2)/2 = qL/4$$

$N_2(x) = \dfrac{x - x_2}{x_3 - x_2}$



The contribution to L_1

$$\int_{x_L}^{x_R} qN_2(x)\,dx =:_R = x_3$$

$$q(x_R - x_L)/2 = q(x_3 - x_2)/2 = qL/4$$

6

Panel 7

Now all the contributions from the elements to the load vector components will be added up to yield

Element 1 Element 2

$$\boldsymbol{L} = \begin{pmatrix} qL/4 + qL/4 \\ qL/4 \end{pmatrix} = \begin{pmatrix} qL/2 \\ qL/4 \end{pmatrix}$$

Element 2

7

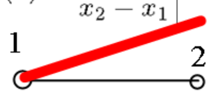
Panel 8

The components of the stiffness matrix are computed in much the same manner.

Loop over elements *e*
 Add contribution to component *ii,ij,jj*, of the stiffness matrix from element *e*

Element 1: $x_L = x_1, x_R = x_2$

$$N_1(x) = \frac{x - x_1}{x_2 - x_1}$$



Only one test function is nonzero over this element. LHS node is associated with prescribed displacement.

The contribution to K_{11}

$$\int_{x_L}^{x_R} N_1'(x) P N_1'(x)\, dx = \int_{x_L}^{x_R} \frac{1}{x_R - x_L} P \frac{1}{x_R - x_L}\, dx = (L/2)\frac{P}{(L/2)^2} = \frac{2P}{L}$$

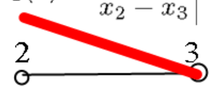
8

Panel 9

There is no contribution to components 12, 22 since the second basis function is zero over element 1.

Element 2: $x_L = x_2, x_R = x_3$

$$N_1(x) = \frac{x - x_3}{x_2 - x_3}$$

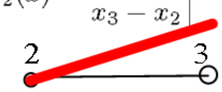


The contribution to K_{11}

$$\int_{x_L}^{x_R} N_1'(x) P N_1'(x)\, dx = \int_{x_L}^{x_R} \frac{1}{x_R - x_L} P \frac{1}{x_R - x_L}\, dx$$

$$= (L/2)\frac{P}{(L/2)^2} = \frac{2P}{L}$$

$$N_2(x) = \frac{x - x_2}{x_3 - x_2}$$



The contribution to $K_{12} = K_{21}$

$$\int_{x_L}^{x_R} N_1'(x) P N_2'(x)\, dx = \int_{x_L}^{x_R} \frac{1}{x_L - x_R} P \frac{1}{x_R - x_L}\, dx = (L/2)\frac{P}{-(L/2)^2} = \frac{-2P}{L}$$

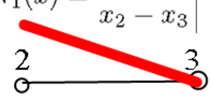
9

Panel 10

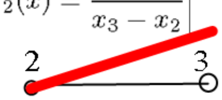
The contribution to K_{22}

$$\int_{x_L}^{x_R} N_2'(x) P N_2'(x)\, dx = \int_{x_L}^{x_R} \frac{1}{x_R - x_L} P \frac{1}{x_R - x_L}\, dx = (L/2)\frac{P}{(L/2)^2} = \frac{2P}{L}$$

$$N_1(x) = \frac{x - x_3}{x_2 - x_3}$$



$$N_2(x) = \frac{x - x_2}{x_3 - x_2}$$



10

Panel 11

Now all the contributions from the elements to the stiffness matrix components will be added up to yield

Element 1 Element 2

$$\boldsymbol{K} = \begin{pmatrix} 2\frac{P}{L} + 2\frac{P}{L} & -2\frac{P}{L} \\ -2\frac{P}{L} & 2\frac{P}{L} \end{pmatrix}$$

Element 2

So you can see that the stiffness matrix and the load vector have the same components as before. The solution has the same values for the deflections at the nodes, except that they are numbered w_1, w_2

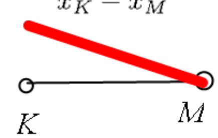
11

Panel 12

We can do more to streamline the computational procedure.The key is to compute the so-called elementwise stiffness matrix and load vector, and then use the so-called assembly procedure.

To compute the elementwise quantities, we shall consider a generic element with nodes at locations x_K, x_M

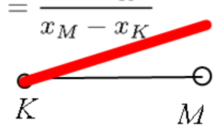
$$N_K(x) = \frac{x - x_M}{x_K - x_M}$$



The element load vector is defined as

$$\boldsymbol{L}^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} qN_K(x)\, dx \\ \int_{x_K}^{x_M} qN_M(x)\, dx \end{pmatrix}$$

$$N_M(x) = \frac{x - x_K}{x_M - x_K}$$



For uniform load this works out to

$$\boldsymbol{L}^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} qN_K(x)\, dx \\ \int_{x_K}^{x_M} qN_M(x)\, dx \end{pmatrix} = q(x_M - x_K)/2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

12

Panel 13

The element stiffness matrix is defined as

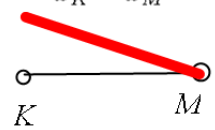
$$\boldsymbol{K}^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} N_K'(x) P N_K'(x)\, dx & \int_{x_K}^{x_M} N_K'(x) P N_M'(x)\, dx \\ \int_{x_K}^{x_M} N_M'(x) P N_K'(x)\, dx & \int_{x_K}^{x_M} N_M'(x) P N_M'(x)\, dx \end{pmatrix}$$

We compute

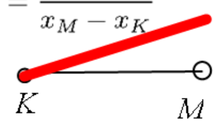
$$\int_{x_K}^{x_M} N_K'(x) P N_K'(x)\, dx = \frac{P}{x_M - x_K}$$

$$\int_{x_K}^{x_M} N_K'(x) P N_M'(x)\, dx = \frac{-P}{x_M - x_K}$$

$$N_K(x) = \frac{x - x_M}{x_K - x_M}$$



$$N_M(x) = \frac{x - x_K}{x_M - x_K}$$



and so the elementwise stiffness matrix is

$$\boldsymbol{K}^{(e)} = \frac{P}{x_M - x_K} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

13

Panel 14

To compute the stiffness matrix for the third time, we will loop over the elements, compute the element stiffness matrix, and assemble it into the global stiffness matrix for the entire structure.

Initially, the global stiffness matrix is empty (zero matrix).

$$\boldsymbol{K} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Element 1: $x_K = x_1, x_M = x_2|$ $\quad x_M - x_K = (L/2)|$

$$\boldsymbol{K}^{(e)} = \frac{2P}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{K}^{(e)} = \frac{2P}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

(column equation numbers: **3** **1**) ← Equation numbers

14

Panel 15

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{array}{c}\begin{array}{cc}\mathbf{3} & \mathbf{1}\end{array}\\ \left(\begin{array}{cc}1 & -1\\ -1 & 1\end{array}\right)\end{array}\begin{array}{c}\\ \mathbf{3}\\ \mathbf{1}\end{array}$$

Equation numbers

Now ***assemble*** it

$$\boldsymbol{K} = \begin{pmatrix}\frac{2P}{L} & 0\\ 0 & 0\end{pmatrix} \qquad \longleftarrow \qquad \boldsymbol{K} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

15

Panel 16

Element 2: $x_K = x_2, x_M = x_3$ $\quad x_M - x_K = (L/2)$

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{array}{c}\begin{array}{cc}\mathbf{1} & \mathbf{2}\end{array}\\ \left(\begin{array}{cc}1 & -1\\ -1 & 1\end{array}\right)\end{array}\begin{array}{c}\\ \mathbf{1}\\ \mathbf{2}\end{array}$$

Equation numbers

16

Panel 17

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix}$$

Column equation numbers: **1** **2** — Equation numbers

Now ***assemble*** it

$$\boldsymbol{K} = \begin{pmatrix} \frac{2P}{L} + \frac{2P}{L} & -\frac{2P}{L} \\ -\frac{2P}{L} & \frac{2P}{L} \end{pmatrix} \longleftarrow \boldsymbol{K} = \begin{pmatrix} \frac{2P}{L} & 0 \\ 0 & 0 \end{pmatrix}$$

17

Panel 18

To compute the load vector, we will loop over the elements, compute the element load vector, and assemble it into the global load vector for the entire structure.

Initially, the global load vector is empty (zero matrix).

$$\boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 1: $x_K = x_1, x_M = x_2$ $x_M - x_K = (L/2)$

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

Equation numbers

18

Panel 19

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\left(\begin{array}{c}1\\1\end{array}\right)\Bigg| \begin{array}{c}\mathbf{3}\\\mathbf{1}\end{array}$$

Equation numbers

Now ***assemble*** it

$$\boldsymbol{L} = \left(\begin{array}{c}\frac{qL}{4}\\0\end{array}\right) \longleftarrow \boldsymbol{L} = \left(\begin{array}{c}0\\0\end{array}\right)$$

19

Panel 20

Element 2: $x_K = x_2, x_M = x_3$ $\quad x_M - x_K = (L/2)$

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\left(\begin{array}{c}1\\1\end{array}\right)$$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\left(\begin{array}{c}1\\1\end{array}\right)\Bigg| \begin{array}{c}\mathbf{1}\\\mathbf{2}\end{array}$$

Equation numbers

20

Panel 21

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix}$$

Equation numbers

Now ***assemble*** it

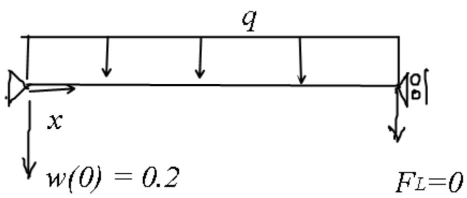
$$\boldsymbol{L} = \begin{pmatrix} \frac{qL}{4} + \frac{qL}{4} \\ \frac{qL}{4} \end{pmatrix} \longleftarrow \boldsymbol{L} = \begin{pmatrix} \frac{qL}{4} \\ 0 \end{pmatrix}$$

21

Panel 1

Exercise 28-c

Extend the problem described in section 3.2 by prescribed support settlement at the left-hand side pin. Solve by hand using the technique of partitioned global system.



1

Panel 2

Here is the definition of the mesh.

Element	Nodes
1	1,2
2	2,3

We will assign the following numbering to the nodal displacements. We start with the displacements that are unknown (sometimes we say free), and then we follow with the displacements that are known (sometimes we say prescribed).

Node	Unknown #
1	3
2	1
3	2

The numbers of the unknowns will determine the numbering of the basis functions.

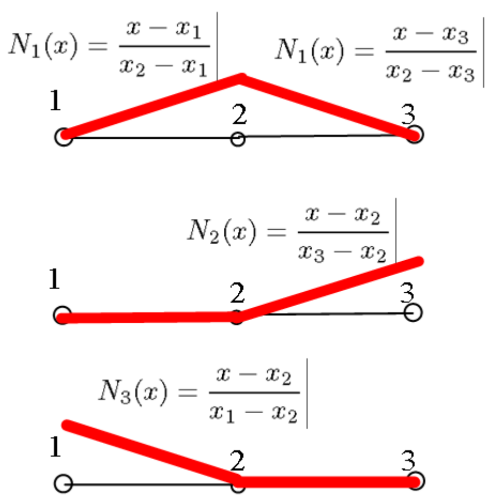
2

Panel 3

The basis functions (= test functions)

The locations of the nodes

i	x_i
1	0
2	L/2
3	L



Note that we are including the third basis function so that we can compute the load and stiffness.

By convention we draw the function above the x-axis when it is positive.

3

Panel 4

Element 1

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Column equation numbers: **3**, **1**; row equation numbers: **3**, **1** (Equation numbers)

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Row equation numbers: **3**, **1** (Equation numbers)

Element 2

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Column equation numbers: **1**, **2**; row equation numbers: **1**, **2** (Equation numbers)

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Row equation numbers: **1**, **2** (Equation numbers)

Global stiffness and load vector

$$\boldsymbol{K} = \frac{P}{L}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \qquad \boldsymbol{L} = \begin{pmatrix} \frac{qL}{2} \\ \frac{qL}{4} \\ \frac{qL}{4} + F_0 \end{pmatrix}$$

Displacement

$$\boldsymbol{d} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

4

Panel 5

The global system of equations is written as

$$\frac{P}{L}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \frac{qL}{2} \\ \frac{qL}{4} \\ \frac{qL}{4} + F_0 \end{pmatrix}$$

= unknown

As the partitioning by the red lines indicates, the global system may be broken up into two parts:

The first part can be used to solve for the unknown displacements

$$\frac{P}{L}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -\frac{P}{L}\begin{pmatrix} -2 \\ 0 \end{pmatrix} w_3 + \begin{pmatrix} \frac{qL}{2} \\ \frac{qL}{4} \end{pmatrix}$$

Support settlement load

Distributed load

while the second part can be used to solve for the reactions

$$F_0 = \frac{P}{L}\begin{pmatrix} -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \frac{qL}{4}$$

5

Panel 6

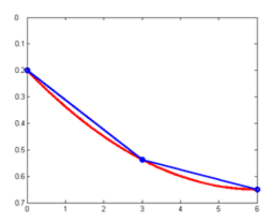
The displacement due to the distributed load of only was already obtained in exercise 28-a...

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{qL^2}{P}\begin{pmatrix} 3/8 \\ 1/2 \end{pmatrix} + \begin{pmatrix} w_3 \\ w_3 \end{pmatrix}$$

...to which we add the contribution of the support settlement

The reaction at $x=0$ is easily obtained as

$$F_0 = \frac{P}{L}(-2)\left(\frac{qL^2}{P}(3/8) + w_3\right) + \frac{P}{L}(2)w_3 - \frac{qL}{4} = -qL$$

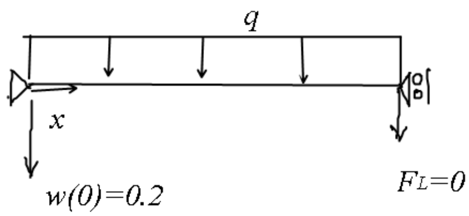


6

Panel 1

Exercise 28-d

Extend the problem described in section 3.2 by prescribed support settlement at the left-hand side pin. Solve by hand using the technique of elementwise support-settlement loads.



1

Panel 2

Here is the definition of the mesh.

Element	Nodes
1	1,2
2	2,3

We will assign the following numbering to the nodal displacements. We start with the displacements that are unknown (sometimes we say free), and then we follow with the displacements that are known (sometimes we say prescribed).

Node	Unknown #
1	3
2	1
3	2

The numbers of the unknowns will determine the numbering of the basis functions.

2

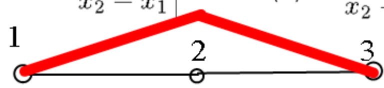
Panel 3

The basis functions (= test functions)

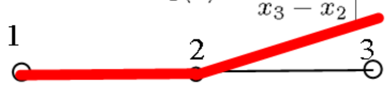
The locations of the nodes

i	*x_i*
1	*0*
2	*L/2*
3	*L*

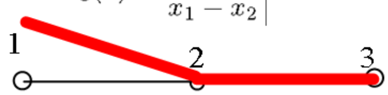
$N_1(x) = \left.\frac{x - x_1}{x_2 - x_1}\right|$ $N_1(x) = \left.\frac{x - x_3}{x_2 - x_3}\right|$



$N_2(x) = \left.\frac{x - x_2}{x_3 - x_2}\right|$



$N_3(x) = \left.\frac{x - x_2}{x_1 - x_2}\right|$



Note that we are including the third basis function so that we can compute the load and stiffness.

By convention we draw the function above the x-axis when it is positive.

3

Panel 4

The global stiffness matrix is assembled as an exercise 28-b from elementwise stiffness matrices.

Element 1: $x_K = x_1, x_M = x_2|$ $x_M - x_K = (L/2)|$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Column equation numbers: **3**, **1**; row equation numbers: **3**, **1** — Equation numbers

Element 2: $x_K = x_2, x_M = x_3|$ $x_M - x_K = (L/2)|$

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Column equation numbers: **1**, **2**; row equation numbers: **1**, **2** — Equation numbers

4

Panel 5

The Elementwise loads due to the distributed load are also assembled as before.

Element 1: $x_K = x_1, x_M = x_2|$ $x_M - x_K = (L/2)|$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

← Equation numbers

Element 2: $x_K = x_2, x_M = x_3|$ $x_M - x_K = (L/2)|$

The elements of the elementwise stiffness matrix are assembled using the equation numbers associated with the nodes.

$$\boldsymbol{L}^{(e)} = \frac{qL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix}$$

← Equation numbers

5

Panel 6

The support-settlement loads are the novelty here. From exercise 28-c we see that the prescribed displacements multiply columns of the global stiffness matrix which are moved on to the right-hand side as loads. Therefore, we note that we can do this multiplication directly on the element stiffness matrices, and assemble the resulting load vectors.

Element 1: $x_K = x_1, x_M = x_2|$ $x_M - x_K = (L/2)|$

The elementwise stiffness matrix is multiplied by w_1 as an unknown displacement, and by w_3 as a prescribed displacements.

$$\begin{matrix} & w_3 & \\ & \mathbf{3} & \mathbf{1} \end{matrix}$$

$$\boldsymbol{K}^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

← Equation numbers

6

Panel 7

The first column will become the load (with a negative sign -- we are moving it onto the right hand side)

w_3

$$\boldsymbol{K}^{(e)} = \frac{2P}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

Columns: **3**, **1** — Equation numbers

$$\boldsymbol{L}^{(e)} = -\frac{2P}{L} \begin{pmatrix} 1 \\ -1 \end{pmatrix} w_3 \quad \begin{matrix} \mathbf{3} \\ \mathbf{1} \end{matrix}$$

Equation numbers

Element 2: No displacement on element 2 is prescribed -- there is no support settlement load generated on this element.

7

Panel 8

In this way we arrive at the same global system of equations as in exercise 28-b

$$\frac{P}{L} \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -\frac{P}{L} \begin{pmatrix} -2 \\ 0 \end{pmatrix} w_3 + \begin{pmatrix} \frac{qL}{2} \\ \frac{qL}{4} \end{pmatrix}$$

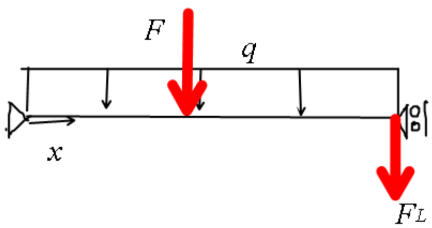
Support settlement load   Distributed load

8

Panel 1

Exercise 28-e

Formulate the boundary value problem for the prestressed cable so that it would allow for intermediate concentrated forces.



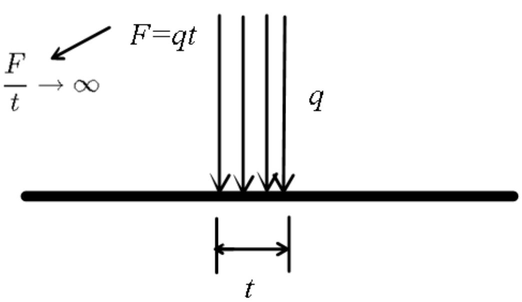
1

Panel 2

The reason we need to modify the deformation of the boundary value problem is that under a concentrated force the second derivative of the deflection (that is the curvature of the cable) is infinite.

We can easily convince ourselves that this is the case if we consider the concentrated force to be the limit of an infinitely narrow distributed load, whose magnitude is adjusted to generate a given nonzero force.

$$Pw'' + q = 0 \qquad q = \frac{F}{t} \to \infty$$



2

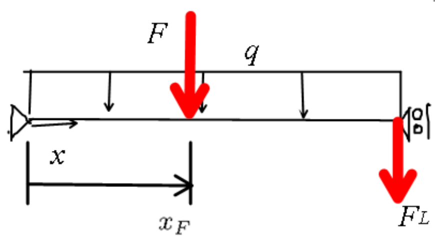
Panel 3

Therefore, since we must avoid taking the second derivative at the location of the concentrated force, we will assume that the equation of the equilibrium holds everywhere in between concentrated forces. Consequently, the length of the cable needs to be divided into segments in between the locations of the concentrated forces. For simplicity we assume there is only one concentrated force present, but the derivation will be applicable also for a different number of concentrated forces.

The Differential equilibrium equation holds in the two intervals

$$Pw'' + q = 0 \qquad 0 \le x < x_F$$

$$x_F < x < L$$



3

Panel 4

The point $x = x_F$ is then as special as the boundaries of the cable. As with the boundary of the cable, the differential equation of equilibrium does not hold at that point, and additional equations are required in order to construct a solution of the boundary value problem.

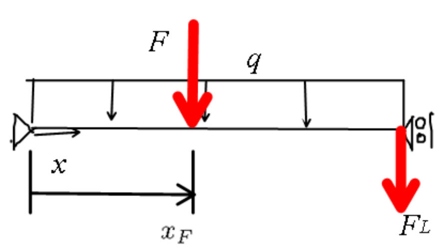
Since we've broken up the original cable length into two pieces, each of which has the boundary consisting of two points, we would expect to have to write down to equations at the additional "boundary point" $x = x_F$

The obvious equation is the continuity of the deflection.

$$w(x_F^-) = w(x_F^+)$$

Here by x_F^- we mean immediately to the left of x_F

x_F^+ we mean immediately to the right of x_F

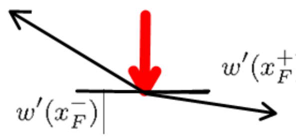


4

Panel 5

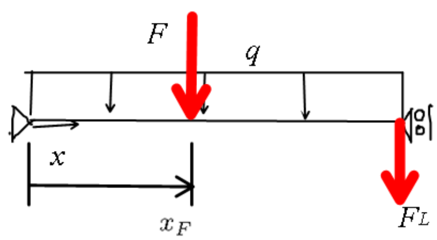
Continuity of slope does not hold, however. The reason is the presence of the concentrated force which needs to balance with tensile forces in the cable.

Writing that equilibrium of an infinitesimally small segment including the point of application of the concentrated force gives



$$Pw'(x_F^+) + F - Pw'(x_F^-) = 0$$

Note that this equation very much resembles the natural boundary condition (1.3). There is nothing accidental about this resemblance, and intermediate forces are treated exactly as boundary conditions in the finite element model.



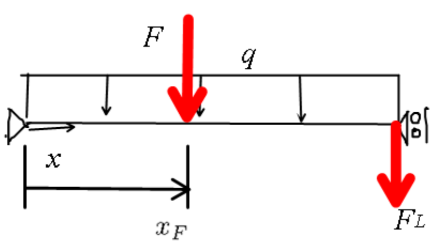
5

Panel 6

The boundary value problem is defined as:

The differential equilibrium equation holds in the two intervals

$$Pw'' + q = 0 \qquad 0 \le x < x_F$$
$$x_F < x < L$$



We have the two boundary conditions

$$w(0) = 0 \qquad F_L - Pw'(L) = 0$$

and the two continuity conditions

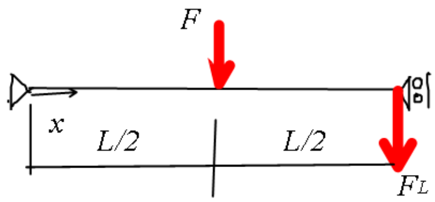
$$w(x_F^-) = w(x_F^+) \qquad Pw'(x_F^+) + F - Pw'(x_F^-) = 0$$

6

Panel 1

Exercise 28-f

Solve by hand the boundary value problem for the prestressed cable using a mesh of two L2 finite elements.



1

Panel 2

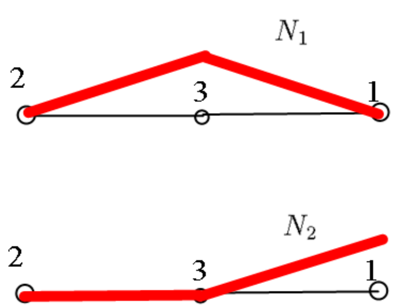
We change a little bit the mesh with the respect to the exercise 28-b.

The locations of the nodes

i	*x_i*
1	*L*
2	*0*
3	*L/2*

Element	Nodes
1	2,3
2	3,1

Node	Equation #
1	1
2	3
3	2



2

Panel 3

The global stiffness matrix is assembled from elementwise stiffness matrices as

Element 1

$$K^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{(rows/columns: 3, 2)}$$

Element 2

$$K^{(e)} = \frac{2P}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{(rows/columns: 2, 1)}$$

Global stiffness matrix

$$K = \frac{P}{L}\begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

The global load vector is assembled directly from the applied forces. Force F_L is applied to node 1 (equation #1), force F is applied to node 3 (equation #2).

$$L = \begin{pmatrix} F_L \\ F \end{pmatrix}$$

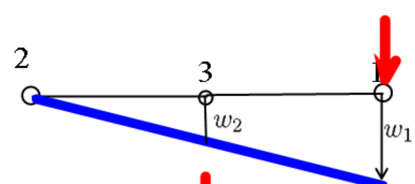
3

Panel 4

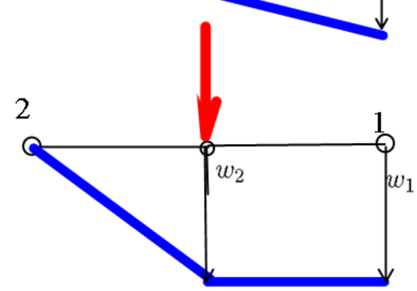
The displacement is $d = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{L}{P}\begin{pmatrix} F_L + F/2 \\ F_L/2 + F/2 \end{pmatrix}$

It will be instructive to consider the results in terms of a superposition.

First consider $F_L \neq 0, F = 0$

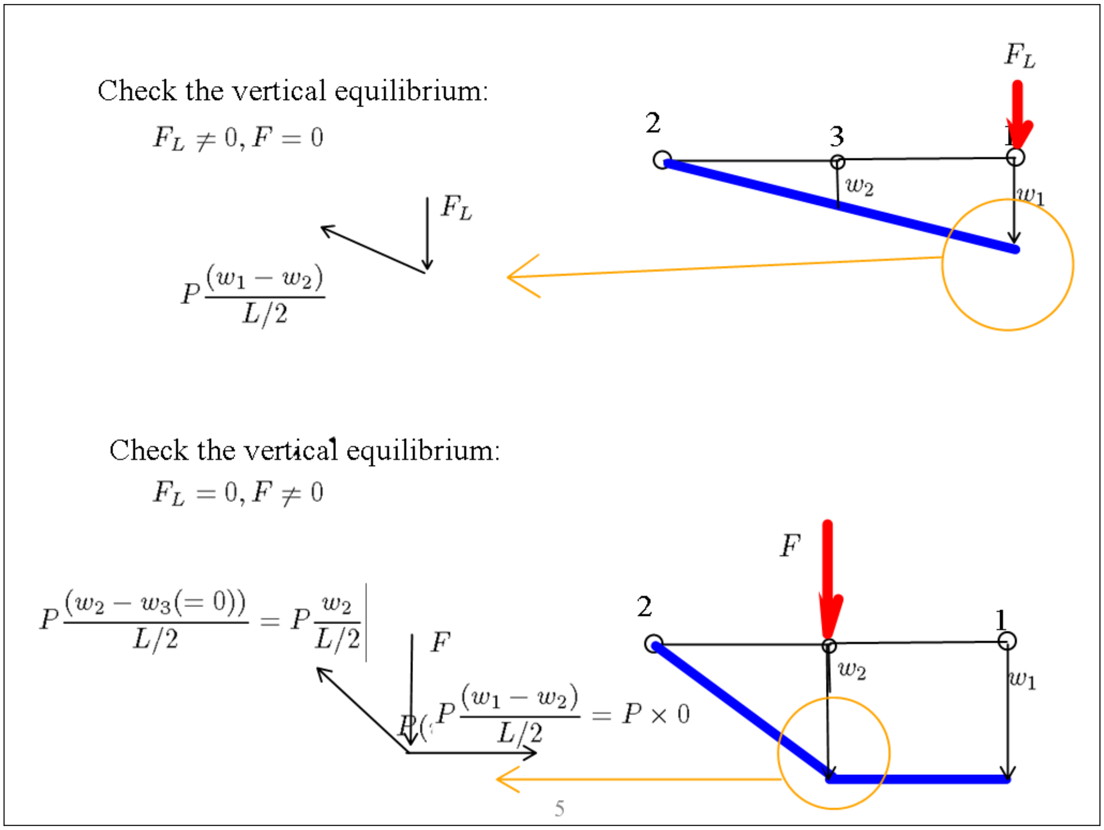


Next, consider $F_L = 0, F \neq 0$



4

Panel 5


Check the vertical equilibrium:
$F_L \neq 0, F = 0$
F_L
$P\frac{(w_1 - w_2)}{L/2}$
2
3
1
w_2
w_1
Check the vertical equilibrium:
$F_L = 0, F \neq 0$
F
$P\frac{(w_2 - w_3(=0))}{L/2} = P\frac{w_2}{L/2}$
$P\frac{(w_1 - w_2)}{L/2} = P \times 0$
5

Panel 1

Exercise 30-a

Demonstrate the sparseness of the stiffness matrix constructed for a mesh of seven L2 finite elements.

1

Panel 2

The global stiffness matrix is assembled from elementwise stiffness matrices as

Element 1

$$\begin{matrix} & \mathbf{3} & \mathbf{2} & \\ \boldsymbol{K}^{(e)} = \frac{2P}{L} & \left(\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \right) & & \begin{matrix} \mathbf{3} \\ \mathbf{2} \end{matrix} \end{matrix}$$

Element 2

$$\begin{matrix} & \mathbf{2} & \mathbf{1} & \\ \boldsymbol{K}^{(e)} = \frac{2P}{L} & \left(\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \right) & & \begin{matrix} \mathbf{2} \\ \mathbf{1} \end{matrix} \end{matrix}$$

Global stiffness matrix

$$\boldsymbol{K} = \frac{P}{L} \left(\begin{matrix} 2 & -2 \\ -2 & 4 \end{matrix} \right)$$

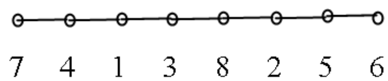
The global load vector is assembled directly from the applied forces. Force F_L is applied to node 1 (equation #1), force F is applied to node 3 (equation #2).

$$\boldsymbol{L} = \left(\begin{matrix} F_L \\ F \end{matrix} \right)$$

2

Panel 3

First we will use a random numbering up the nodes. The equation numbers will be taken the same as the numbers of the nodes. (Note that we are ignoring the boundary conditions: if there were prescribed displacements at the ends of the cable, we would number those nodes last.)



The element stiffness matrix is for all elements the same

$$\boldsymbol{K}^{(e)} = \frac{P}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

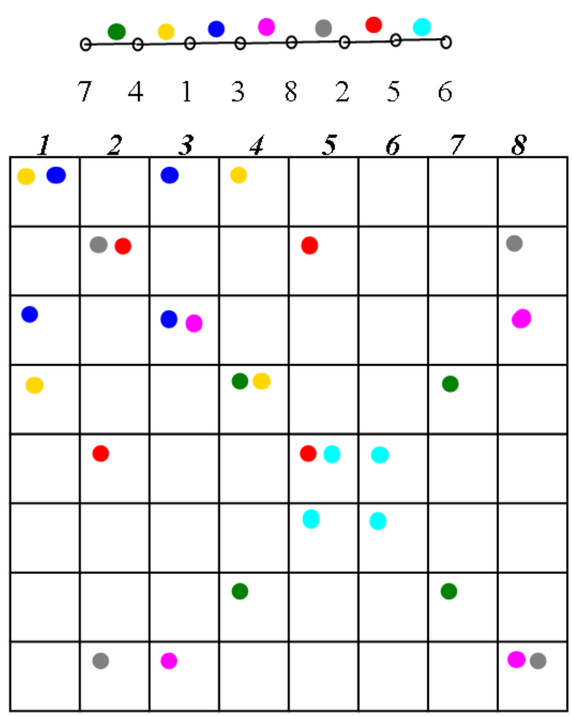
Here h is the element length, which is the same for all elements.

3

Panel 4

The elements are indicated by color. Their 2x2 stiffness matrices are assembled using the equation numbers.

Note that the stiffness matrix is very sparse: wherever a box is empty it holds a zero.



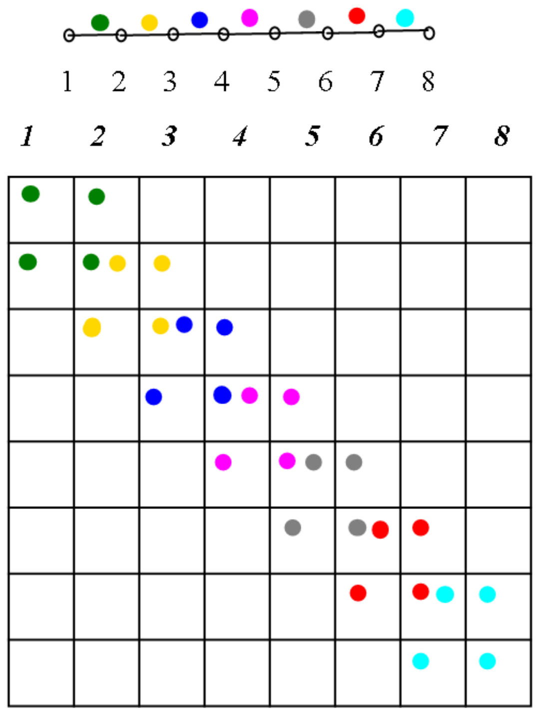
4

Panel 5

Now we are going to switch to a natural numbering, left-to-right.

Here the structure of the stiffness matrix is as good as it gets (tri-diagonal).

Matrices of this nature are called ***banded*** since the non-zeros occur only in a band along the diagonal.

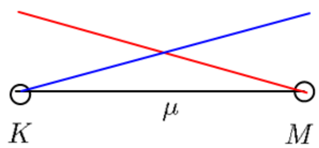


5

Panel 1

Exercise 32-a

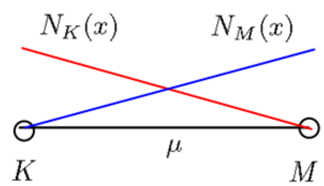
Compute the elementwise mass matrix for the L2 finite element using one-point Gaussian quadrature.



1

Panel 2

The mass matrix elements that represent the elementwise interactions of the test function and the trial basis functions are represented by this matrix:



$$M^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} N_K(x)\mu N_K(x)\,dx & \int_{x_K}^{x_M} N_K(x)\mu N_M(x)\,dx \\ \int_{x_K}^{x_M} N_M(x)\mu N_K(x)\,dx & \int_{x_K}^{x_M} N_M(x)\mu N_M(x)\,dx \end{pmatrix}$$

For simplicity we can assume here that the mass density is constant on each finite element.

2

Panel 3

The quadrature formula (2.26) for the one-point Gaussian quadrature uses the following table

k	ξ_k	W_k
1	0	2

For instance, we have

$$\int_{x_K}^{x_M} N_M(x)\mu N_K(x)\,dx \approx \sum_k \underbrace{(N_M(\xi_k)\mu N_K(\xi_k))}_{\text{Integrand}} \underbrace{\left(\frac{x_M - x_K}{2}\right)}_{\text{Jacobian}} \underbrace{(W_k)}_{\text{Weight}}$$

$$= (1/2)\mu(1/2)\left(\frac{x_M - x_K}{2}\right) 2 = \mu\frac{x_M - x_K}{4}$$

The elementwise mass matrix obtained from one-point Gaussian quadrature

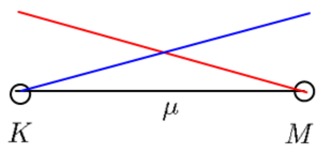
$$\boldsymbol{M}^{(e)} = \mu\frac{x_M - x_K}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

3

Panel 1

Exercise 32-b

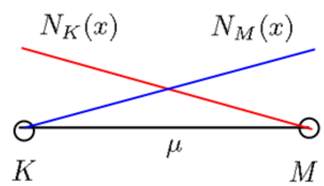
Compute the elementwise mass matrix for the L2 finite element using Simpson's quadrature.



1

Panel 2

The mass matrix elements that represent the elementwise interactions of the test function and the trial basis functions are represented by this matrix:



$$\boldsymbol{M}^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} N_K(x)\mu N_K(x)\, dx & \int_{x_K}^{x_M} N_K(x)\mu N_M(x)\, dx \\ \int_{x_K}^{x_M} N_M(x)\mu N_K(x)\, dx & \int_{x_K}^{x_M} N_M(x)\mu N_M(x)\, dx \end{pmatrix}$$

For simplicity we can assume here that the mass density is constant on each finite element.

2

Panel 3

The quadrature formula (2.26) for the Simpson's quadrature uses the following table

k	ξ_k	W_k
1	-1	1/3
2	0	4/3
3	+1	1/3

We have

$$\int_{x_K}^{x_M} N_M(x)\mu N_K(x)\,dx \approx \sum_k \underbrace{(N_M(\xi_k)\mu N_K(\xi_k))}_{\text{Integrand}} \underbrace{\left(\frac{x_M - x_K}{2}\right)}_{\text{Jacobian}} \underbrace{(W_k)}_{\text{Weight}}$$

Note that for *k=1* and *k=3* we have one of the basis functions in the product be equal to zero. Therefore, for this case the quadrature formula gives

$$= (1/2)\mu(1/2)\left(\frac{x_M - x_K}{2}\right)4/3 := \mu\frac{x_M - x_K}{6}$$

3

Panel 4

For the diagonal elements we have for instance

$$\int_{x_K}^{x_M} N_K(x)\mu N_K(x)\,dx \approx$$

$$= (1)\mu(1)\left(\frac{x_M - x_K}{2}\right)1/3 + (1/2)\mu(1/2)\left(\frac{x_M - x_K}{2}\right)4/3 + (0)\mu(0)\left(\frac{x_M - x_K}{2}\right)1/3$$

$$= \mu\frac{x_M - x_K}{3}$$

and the same result is obtained for $\int_{x_K}^{x_M} N_M(x)\mu N_M(x)\,dx$

As a result, the elementwise mass matrix is obtained as

$$\boldsymbol{M}^{(e)} = \mu\frac{x_M - x_K}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

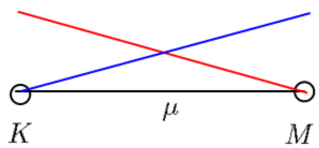
Since the product of two piecewise linear basis functions is a quadratic function, the Simpson's rule will be able to give us an exact integration, so this mass matrix coincides with the analytically integrated answer.

4

Panel 1

Exercise 32-c

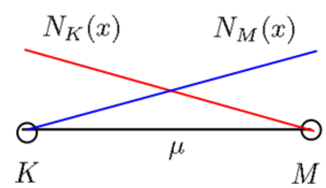
Compute the elementwise mass matrix for the L2 finite element using trapezoidal-rule quadrature.



1

Panel 2

The mass matrix elements that represent the elementwise interactions of the test function and the trial basis functions are represented by this matrix:



$$\boldsymbol{M}^{(e)} = \begin{pmatrix} \int_{x_K}^{x_M} N_K(x)\mu N_K(x)\,dx & \int_{x_K}^{x_M} N_K(x)\mu N_M(x)\,dx \\ \int_{x_K}^{x_M} N_M(x)\mu N_K(x)\,dx & \int_{x_K}^{x_M} N_M(x)\mu N_M(x)\,dx \end{pmatrix}$$

For simplicity we can assume here that the mass density is constant on each finite element.

2

Panel 3

The quadrature formula (2.26) for the Simpson's quadrature uses the following table

k	ξ_k	W_k
1	-1	1
2	+1	1

We have

$$\int_{x_K}^{x_M} N_M(x)\mu N_K(x)\,dx \approx \sum_k \underbrace{(N_M(\xi_k)\mu N_K(\xi_k))}_{\text{Integrand}} \underbrace{\left(\frac{x_M - x_K}{2}\right)}_{\text{Jacobian}} \underbrace{(W_k)}_{\text{Weight}}$$

Note that for both *k=1* and *k=2* we have one of the basis functions in the product be equal to zero. Therefore, for this case the quadrature formula gives

$$= 0$$

3

Panel 4

For the diagonal elements we have

$$\int_{x_K}^{x_M} N_K(x)\mu N_K(x)\,dx \approx$$

$$= (1)\mu(1)\left(\frac{x_M - x_K}{2}\right)1 + (0)\mu(0)\left(\frac{x_M - x_K}{2}\right)1 = \mu\frac{x_M - x_K}{2}$$

and the same result is obtained for $\int_{x_K}^{x_M} N_M(x)\mu N_M(x)\,dx$

As a result, the elementwise mass matrix is obtained as

$$\boldsymbol{M}^{(e)} = \mu\frac{x_M - x_K}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

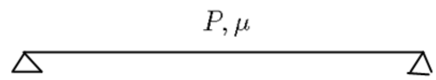
Note that the elementwise mass matrix is ***diagonal.*** Assembling diagonal mass matrices into the global mass matrix will make also the global mass matrix diagonal. Some solution techniques derive their efficiency from the mass matrix being diagonal (the so-called explicit time-stepping).

4

Panel 1

Exercise 33-a

Find analytically the free vibration modes and frequencies for a simply supported cable with uniform mass density.



1

Panel 2

The Initial Boundary Value Problem (IBVP) is written as

$$Pw'' = \mu\ddot{w} \qquad w(0) = 0, w(L) = 0$$

We will use the technique of separation: the displacement function will be sought as a product that separates space and time

$$w(x,t) = \phi(x)\psi(t)$$

The first function describes the ***shape*** of the cable, the second function gives it a ***variation in time***.

Substituting, we obtain

$$P\phi''(x)\psi(t) = \mu\phi(x)\ddot{\psi}(t)$$

When this is rewritten as

$$(P/\mu)\phi''(x)/\phi(x) = \ddot{\psi}(t)/\psi(t)$$

we realize that the ratios of the functions on either side must be constants since on the left-hand side we have a function of the space coordinate, while on the right-hand side we have a function of the time.

2

Panel 3

Functions whose second derivatives are proportional to themselves are the exponentials. Therefore we shall assume

$$\phi(x) = A\exp(\lambda x) \qquad \psi(t) = B\exp(\beta t)$$

where the constants may be in general complex. (But the functions themselves must come out real; we will use this fact presently.)

Upon substitution into

$$(P/\mu)\phi''(x)/\phi(x) = \ddot{\psi}(t)/\psi(t)$$

we obtain the following relationship between the constants

$$(P/\mu)\lambda^2 = \beta^2$$

Using the Euler identity $\exp(\lambda x) = \exp(\mathrm{Re}\lambda x)\left(\cos(\mathrm{Im}\lambda x) + \mathrm{i}\sin(\mathrm{Im}\lambda x)\right)$

we can write for the function that describes the shape of the cable

$$\phi(x) = A\exp(\lambda x) = (\mathrm{Re}A + \mathrm{i}\mathrm{Im}A)\exp(\mathrm{Re}\lambda x)\left(\cos(\mathrm{Im}\lambda x) + \mathrm{i}\sin(\mathrm{Im}\lambda x)\right)$$

3

Panel 4

which simplifies to

$$\phi(x) = \exp(\mathrm{Re}\lambda x)\left(\mathrm{Re}A\cos(\mathrm{Im}\lambda x) - \mathrm{Im}A\sin(\mathrm{Im}\lambda x)\right)$$

since the resulting function must be real.

By inspection of the boundary conditions it is clear that only the sine function is admissible. It satisfies the boundary condition at $x=0$, and it can also satisfy the boundary condition at $x=L$ if

$$\sin(\mathrm{Im}\lambda L) = 0$$

from where it follows

$$\mathrm{Im}\lambda = k\pi/L \qquad k=1,2,3,\ldots$$

(We discount the possibility of $k=0$: the cable would not deflect at all.)

4

Panel 5

This result is substituted into $(P/\mu)\lambda^2 = \beta^2$
to yield

$$(P/\mu)\lambda^2 = (P/\mu)\left((\mathrm{Re}\lambda)^2 - (\mathrm{Im}\lambda)^2 + 2\mathrm{i}\mathrm{Re}\lambda\mathrm{Im}\lambda\right) = \beta^2 = \left((\mathrm{Re}\beta)^2 - (\mathrm{Im}\beta)^2 + 2\mathrm{i}\mathrm{Re}\beta\mathrm{Im}\beta\right) \quad (*)$$

At this point we realize that the time-dependence function $\psi(t)$ should represent harmonic (sinusoidal) motion. Therefore we must require that

$$\mathrm{Re}\beta = 0$$

Going back to the (*) equation, it immediately follows that

$$\mathrm{Re}\lambda = 0$$

so that finally we conclude

$$(P/\mu)(\mathrm{Im}\lambda)^2 = (\mathrm{Im}\beta)^2$$

5

Panel 6

As is the convention, we shall call

$$\mathrm{Im}\beta = \omega$$

In this way we arrive at the relationship that defines the natural frequencies of vibration

$$\omega = \frac{k\pi}{L}\sqrt{P/\mu} \qquad k = 1, 2, 3, \ldots$$

Remark: In order to solve the initial boundary value problem completely, we could consider initial conditions in order to determine all the constants involved. Since we are interested in the so-called steady-state free harmonic motion, we do not need the precise time dependence, and for instance taking a cosine time variation is adequate.

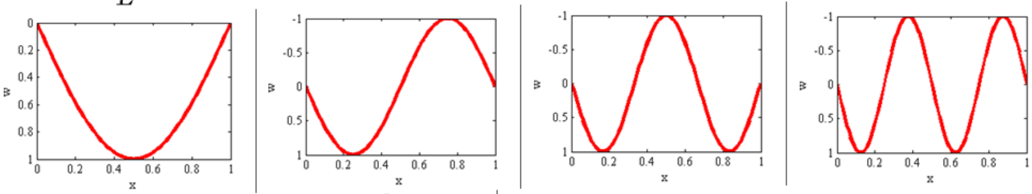
6

Panel 7

Selected mode shapes

$$\omega_1 = \frac{\pi}{L}\sqrt{P/\mu}$$

$$\omega_3 = \frac{3\pi}{L}\sqrt{P/\mu}$$



$$\omega_2 = \frac{2\pi}{L}\sqrt{P/\mu}$$

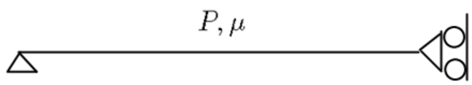
$$\omega_4 = \frac{4\pi}{L}\sqrt{P/\mu}$$

7

Panel 1

Exercise 33-b

Find analytically the free vibration modes and frequencies for a simply-supported/roller cable with uniform mass density.



1

Panel 2

The Initial Boundary Value Problem (IBVP) is written as

$$Pw'' = \mu\ddot{w} \qquad w(0) = 0, w'(L) = 0$$

In the same way as an exercise 33-a we will construct the solution for the displacement as

$$w(x,t) = \phi(x)\psi(t)$$

The next few steps are identical to those of exercise 33-a.

We arrive at

$$\phi(x) = \exp(\mathrm{Re}\lambda x)\left(\mathrm{Re}A\cos(\mathrm{Im}\lambda x) - \mathrm{Im}A\sin(\mathrm{Im}\lambda x)\right)$$

as before. The boundary conditions are different, however.

As before, the sine function is admissible as it satisfies the boundary condition at $x=0$, and it can also satisfy the boundary condition at $x=L$ if

$$w'(L) = \phi'(L) = \exp(\mathrm{Re}\lambda L)\mathrm{Im}A\cos(\mathrm{Im}\lambda L) = 0 \longrightarrow \cos(\mathrm{Im}\lambda L) = 0$$

2

Panel 3

It follows that

$$\mathrm{Im}\lambda = \frac{(k-\frac{1}{2})\pi}{L} \qquad k=1,2,3,\ldots$$

3

Panel 4

This result is substituted into $(P/\mu)\lambda^2 = \beta^2$
and as before we find

$$\mathrm{Re}\beta = 0 \qquad \mathrm{Re}\lambda = 0$$

and

$$(P/\mu)(\mathrm{Im}\lambda)^2 = (\mathrm{Im}\beta)^2$$

As before, we shall call

$$\mathrm{Im}\beta = \omega$$

and we arrive at the relationship that defines the natural frequencies of vibration

$$\omega = \frac{(k-\frac{1}{2})\pi}{L}\sqrt{P/\mu} \quad k = 1,2,3,\ldots$$

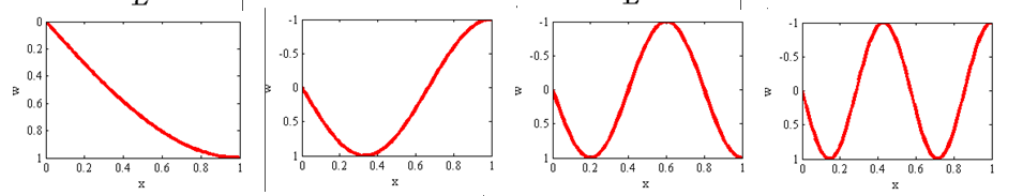
4

Panel 5

Selected Mode Shapes

$$\omega_1 = \frac{\frac{1}{2}\pi}{L}\sqrt{P/\mu}$$

$$\omega_3 = \frac{5\frac{1}{2}\pi}{L}\sqrt{P/\mu}$$



$$\omega_2 = \frac{3\frac{1}{2}\pi}{L}\sqrt{P/\mu}$$

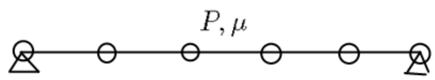
$$\omega_4 = \frac{7\frac{1}{2}\pi}{L}\sqrt{P/\mu}$$

5

Panel 1

Exercise 33-c

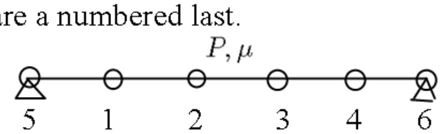
Find the free vibration modes and frequencies for a simply supported cable with uniform mass density using a mesh of five L2 finite elements. Use the trapezoidal integration rule.



1

Panel 2

We shall design the mesh as follows (as an example: we could have chosen a different numbering). Note that we have made sure the nodes associated with supports are a numbered last.



The element length is $h{=}L/5$

The elementwise stiffness matrix is evaluated exactly with the trapezoidal rule.

$$\boldsymbol{K}^{(e)} = \frac{P}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

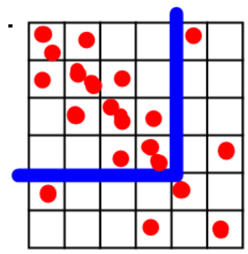
The elementwise mass matrix is diagonal with the trapezoidal rule. (See exercise 32-c.)

$$\boldsymbol{M}^{(e)} = \frac{\mu h}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2

Panel 3

The structure of the global stiffness matrix is easily established by graphically assembling the elementwise stiffness matrix. Each red dot corresponds to (+/-)*P/h*.

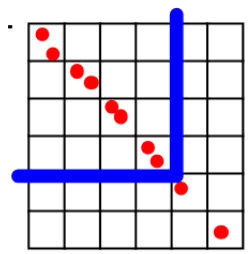


The block corresponding to actual unknowns is indicated in blue.

3

Panel 4

Similarly for the structure of the mass matrix. Each red dot corresponds to *mu*h/2*.



The block corresponding to actual unknowns is indicated in blue.

4

Panel 5

With the mass and stiffness matrix at hand, we can solve the eigenvalue problem

$$\boldsymbol{K}\boldsymbol{\phi} = \omega^2 \boldsymbol{M}\boldsymbol{\phi}$$

We can solve the eigenvalue problem without specifying the tensile force, the length of the cable and the mass density if we use the following trick: defining the matrices

```
>> Kt =[2,-1, 0, 0;
-1, 2,-1, 0;
0, -1, 2,-1;
0, 0, -1, 2];
Mt =diag( [2,2,2,2] );
```

we can write

$$\boldsymbol{K}\boldsymbol{\phi} = \omega^2 \boldsymbol{M}\boldsymbol{\phi} \qquad \frac{P}{h}\,\mathrm{Kt}\ \phi = \frac{\mu h}{2}\,\omega^2\ \mathrm{Mt}\ \phi$$

which means that we can solve the eigenvalue problem

$$\mathrm{Kt}\ \phi = \frac{\mu h^2}{2P}\,\omega^2\ \mathrm{Mt}\ \phi = \rho^2\ \mathrm{Mt}\ \phi$$

Panel 6

This is easily done with Matlab:

```
>> Kt =[2,-1, 0, 0;
-1, 2,-1, 0;
0, -1, 2,-1;
0, 0, -1, 2];
Mt =diag( [2,2,2,2] );
[V,D] =eig(Kt,Mt)

V =
  0.262865556059567  -0.425325404176020  -0.425325404176020  -0.262865556059567
  0.425325404176020  -0.262865556059567   0.262865556059567   0.425325404176020
  0.425325404176020   0.262865556059567   0.262865556059567  -0.425325404176020
  0.262865556059567   0.425325404176020  -0.425325404176020   0.262865556059567


D =

  0.190983005625053                   0                   0                   0
                  0   0.690983005625053                   0                   0
                  0                   0   1.309016994374947                   0
                  0                   0                   0   1.809016994374947
```

Panel 7

This leads to the prediction of the first frequency of free vibration

$$\omega^2 = \frac{2P}{\mu h^2}\ 0.19098300562505$$

$$\omega = \frac{\sqrt{2 \times 25 \times 0.190983005625053}}{L}\sqrt{P/\mu} = \frac{3.090}{L}\sqrt{P/\mu}$$

This may be compared with the analytical prediction

$$\omega = \frac{\pi}{L}\sqrt{P/\mu}$$

7

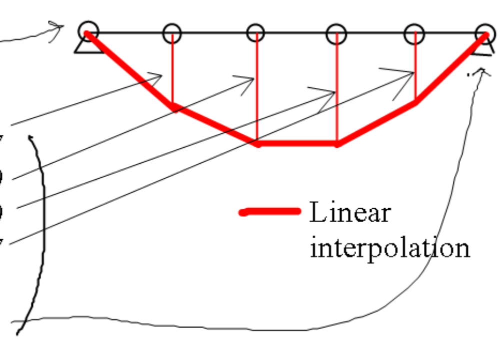
Panel 8

The first mode of vibration is the first column of the matrix V. The meaning of those numbers is elements of the first eigenvector ϕ_1

which may be visualized by forming the linear combination

$$\sum_j N_j(x)(\phi_1)_j$$

By $(\phi_1)_j$ we mean the j-th component of the eigenvector ϕ_1



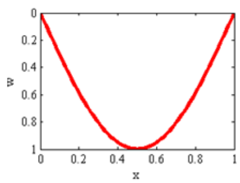
$$(\phi_1) = \begin{pmatrix} 0.262865556059567 \\ 0.425325404176020 \\ 0.425325404176020 \\ 0.262865556059567 \\ 0 \\ 0 \end{pmatrix}$$

8

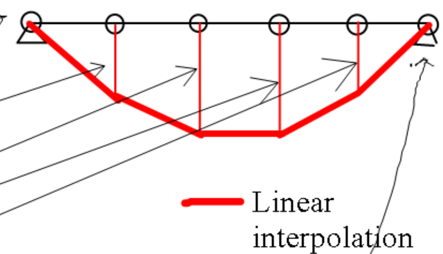
Panel 9

The finite element results may be compared with the analytical mode shape and frequency

$$\omega_1 = \frac{\pi}{L}\sqrt{P/\mu}$$



$$\omega = \frac{\sqrt{2 \times 25 \times 0.190983005625053}}{L}\sqrt{P/\mu} = \frac{3.090}{L}\sqrt{P/\mu}$$



$$(\phi_1) = \begin{pmatrix} 0.262865556059567 \\ 0.425325404176020 \\ 0.425325404176020 \\ 0.262865556059567 \\ 0 \\ 0 \end{pmatrix}$$

9

Panel 1

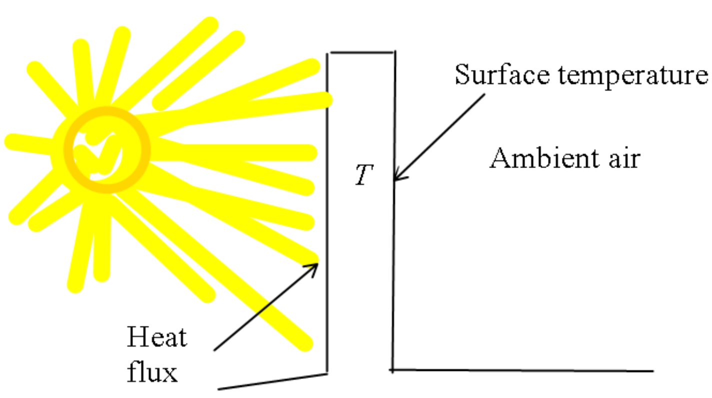
Exercise 56-a

Formulate the boundary conditions for a control volume for heat conduction through the thickness of a wall of large extent (away from the edges).

1

Panel 2

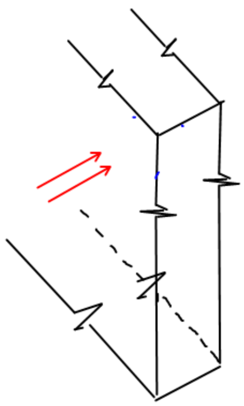
Consider a wall which is large in extent compared to its thickness. Away from the edges of the wall, we can make the observation that the heat flows essentially in the direction of the thickness. For definiteness, take for instance a wall with given heat flux on one side (solar rays), and transfer into ambient air on the other side.



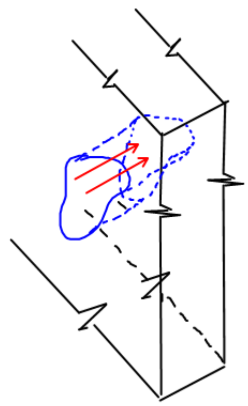
2

Panel 3

Away from the edges of the wall, we can make the observation that the heat flows essentially in the direction of the thickness.



Since the heat energy flows through the thickness, drawing a closed curve on one face and projecting it towards the other face perpendicularly to the plane of the wall creates a kind of imagined "pipe" through which the heat flows.

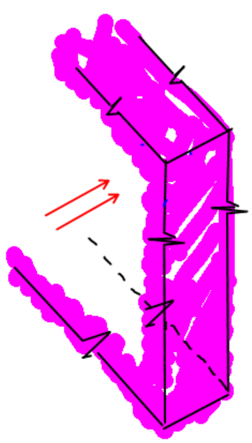


The heat enters one potato-shaped cross-section, travels through the "pipe", and exits the other potato-shaped cross-section. No heat enters or leaves through the cylinder wall (highlighted in green).

3

Panel 4

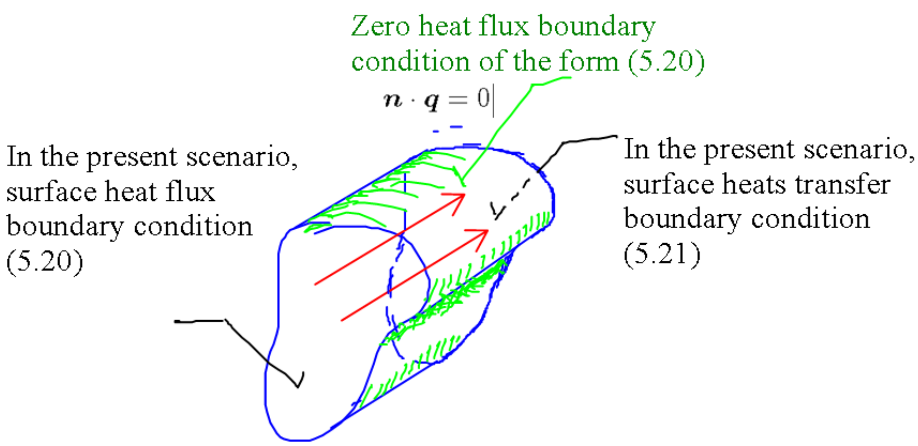
The heat energy flux may be very different around the edges of the wall.



4

Panel 5

Here are the boundary conditions on the imaginary "pipe". Note that in the present scenario we consider some specific boundary conditions on the front and back potato shaped cross-section. These would very from case to case. The boundary condition indicated in green would stay the same.



5

Panel 1

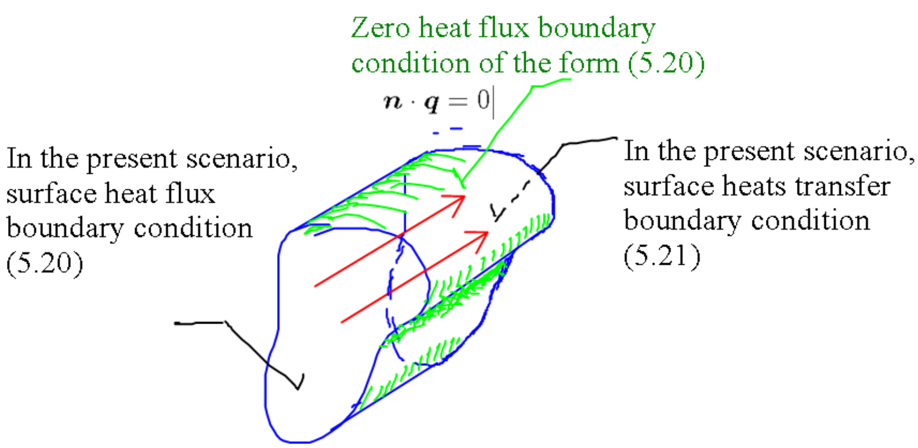
Exercise 59-a

Formulate a one-independent-coordinate Galerkin model for heat conduction through the thickness of a wall of large extent (away from the edges).

1

Panel 2

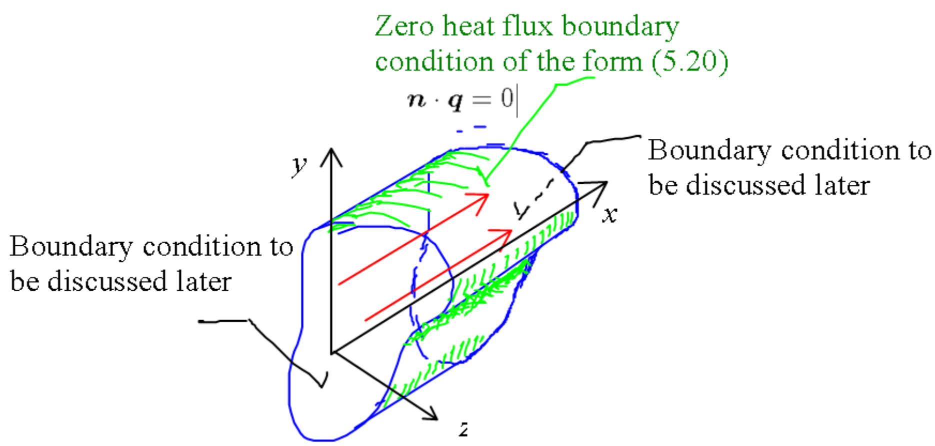
The boundary conditions on the imaginary "pipe". Note that in the present scenario we consider some specific boundary conditions on the front and back potato shaped cross-section. These would very from case to case. The boundary condition indicated in green would stay the same.



2

Panel 3

The boundary conditions on the imaginary "pipe" -- the cylindrical control volume through which the heat flows -- have been discussed in exercise 56-a. Note that the through the thickness coordinate is x, and the coordinates in the plane of the wall are y, z.



3

Panel 4

We will derive the Galerkin model so that it operates with x as the only space coordinate. In other words, everything in the Galerkin weighted residual will be independent of y and z.

In order to achieve this, we will have to require that the boundary conditions on the faces of the wall and the initial conditions throughout the region be independent of the coordinates y and z. For instance, let us assume that at $x=0$ the temperature is prescribed as a function of time.

$$T(x=0,t) = \overline{T}(x=0,t)$$

4

Panel 5

As follows from our physical observation that the heat energy flows only through the thickness of the wall, we must conclude that the temperature field is independent of the coordinates y and z. A consequence is that the gradient components of the temperature in the plane of the wall are identically zero

$$\frac{\partial T}{\partial y} = 0 \qquad \frac{\partial T}{\partial z} = 0 \qquad \text{everywhere}$$

The constitutive equation links the temperature gradients to the heat fluxes. Also the constitutive equation must not depend on y and z. It may depend on the through-the-thickness coordinate. Composite or layered panels may be of this nature.

Since only the x- component should be nonzero, the constitutive equation must also satisfy

$$q_y = 0, q_z = 0$$

5

Panel 6

As we have

$$q_y = -\kappa_{yx}\frac{\partial T}{\partial x} - \kappa_{yy}\frac{\partial T}{\partial y} - \kappa_{yz}\frac{\partial T}{\partial z}$$

zero

$$q_z = -\kappa_{zx}\frac{\partial T}{\partial x} - \kappa_{zy}\frac{\partial T}{\partial y} - \kappa_{zz}\frac{\partial T}{\partial z}$$

zero

the material must satisfy $\kappa_{yx} = 0, \kappa_{zx} = 0$

All isotropic materials (metals, polymers, concrete and such) satisfy this condition, and many orthotropic materials such as composites or layered structures where the layers are parallel to the plane yz would also satisfy this condition.

6

Panel 7

The two cross sectional areas at $x=0$ and $x=L$ (the thickness of the wall is denoted L) may be associated with any type of boundary condition, prescribed temperature, heat flux, or surface heat transfer. For simplicity we will include all three possible boundary conditions, even though there are only two surfaces on which they may be applied. This means that if one type of boundary condition is not present, it should be ignored in the formulation.

The starting point for the actual formulation of the Galerkin weighted residual is equation (6.9)

$$\int_V \eta c_V \frac{\partial T}{\partial t}\, \mathrm{d}V + \int_V (\mathrm{grad}\eta)\, \boldsymbol{\kappa}(\mathrm{grad}T)^T\, \mathrm{d}V - \int_V \eta Q\, \mathrm{d}V$$
$$+ \int_{S_2} \eta\, \overline{q}_n\, \mathrm{d}S + \int_{S_3} \eta\, h(T - T_a)\, \mathrm{d}S = 0, \quad \eta(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \in S_1\,. \tag{6.9}$$

7

Panel 8

Let us take up the first-term: it is a volume integral $\int_V \eta c_V \frac{\partial T}{\partial t}\, dV$

We talked about the thermal conductivity not being a function of y and z. The same goes for c_V

If we define the test function to be independent of y and z, nothing in the above integral will in fact depend on y and z. Therefore, the integrand is constant with the respect to y and z and we can write

$$\int_V \eta c_V \frac{\partial T}{\partial t}\, dV = \int_S dS \int_0^L \eta c_V \frac{\partial T}{\partial t}\, dx = S \int_0^L \eta c_V \frac{\partial T}{\partial t}\, dx$$

where S=the cross-sectional area of the cylindrical control volume.

As only the x-component of the gradients is nonzero, we can also write

$$\int_V (\mathrm{grad}\eta)\, \boldsymbol{\kappa}(\mathrm{grad}T)^T\, dV = S \int_0^L \frac{\partial \eta}{\partial x} \kappa_{xx} \frac{\partial T}{\partial x}\, dx$$

8

Panel 9

Clearly, the same reasoning may be applied to the surface integrals as well. For instance

$$\int_{S_2} \eta\, \overline{q}_n \, dS = S\eta|_{S_2} \, (\overline{q}_n)|_{S_2}$$

Here $\eta|_{S_2}, (\overline{q}_n)|_{S_2}$ are the test function in the prescribed value of the heat flux on the cross-section where the boundary conditions (5.20) is being prescribed.

The Galerkin formulation using a single independent space coordinate is therefore written as

$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\, dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\, dx - S\int_0^L \eta Q\, dx +$$

$$S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\, h(T - T_a)|_{S_3} = 0, \quad \eta|_{S_1} = 0$$

Here by $\eta|_{S_1} = 0$ we mean that the test function must vanish in the cross-section where the temperature is prescribed.

9

Panel 10

Note that we still keep the cross-section area *S* in the weighted residual expression, even though we could have canceled it. The reason is that it allows us to keep in mind that the equation still models the flow of heat energy through a three-dimensional body. Keeping track of the units is also easier with the cross-sectional area in place (all the expressions are in watts!).

10

Panel 1

Exercise 59-b

Compare the Galerkin models for the vibrating cable and the one-independent-coordinate model of heat conduction.

1

Panel 2

The Galerkin formulation of heat conduction using a single independent space coordinate is written as

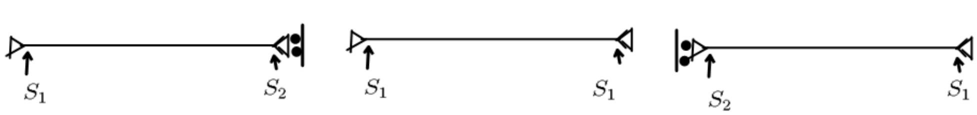
$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\, dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\, dx - S\int_0^L \eta Q\, dx +$$

$$S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\; h(T-T_a)|_{S_3} = 0, \qquad \eta|_{S_1} = 0$$

Rewriting equation (2.11) (mainly changing the signs, and writing the boundary conditions as at points S_1 S_2 in order to allow for supports at either end) yields

$$\int_0^L \eta\mu\ddot{w}\, dx + \int_0^L \frac{\partial \eta}{\partial x}P\frac{\partial w}{\partial x}\, dx - \int_0^L \eta q\, dx - \eta|_{S_2}F|_{S_2} = 0,$$

$$\eta|_{S_1} = 0$$



2

Panel 3

Now we can compare the two formulations term by term:

$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\,dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\,dx - S\int_0^L \eta Q\,dx + S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\, h(T-T_a)|_{S_3} = 0, \quad \eta|_{S_1} = 0$$

$$\int_0^L \eta\mu\ddot{w}\,dx + \int_0^L \frac{\partial\eta}{\partial x}P\frac{\partial w}{\partial x}\,dx - \int_0^L \eta q\,dx - \eta|_{S_2}F|_{S_2} = 0, \quad \eta|_{S_1} = 0$$

Capacity ~ Mass

Conductivity ~ Stiffness

Heat generation ~ Transverse load

Heat flux BC ~ Force BC

Temperature ~ Deflection
Temperature gradient ~ Slope
Heat flux ~ Transverse force

3

Panel 4

Importantly, the term with the derivatives is second-order for the cable, but first order for the heat conduction problem.

Therefore, the heat conduction problem leads to real-exponential like solutions (decay), and the cable problem leads to vibration (oscillations).

For statics (all time derivatives are zero), the two models are very similar, but the heat-conduction model is quite a bit richer in that it allows for the thermal conductivity to be a function of *x*. For the cable the prestress force is a constant.

4

Panel 1

Exercise 59-c

Develop the analogy of the heat surface-transfer boundary condition for the Galerkin model for the vibrating cable.

1

Panel 2

In exercise 59-b we have compared the two formulations term by term:

$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\,dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\,dx - S\int_0^L \eta Q\,dx +$$

$$S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\, h(T - T_a)|_{S_3} = 0, \quad \eta|_{S_1} = 0$$

$$\int_0^L \eta\mu\ddot{w}\,dx + \int_0^L \frac{\partial \eta}{\partial x}P\frac{\partial w}{\partial x}\,dx - \int_0^L \eta q\,dx - \eta|_{S_2}F|_{S_2} = 0,$$

$$\eta|_{S_1} = 0$$

Capacity ~ Mass

Conductivity ~ Stiffness

Heat generation ~ Transverse load

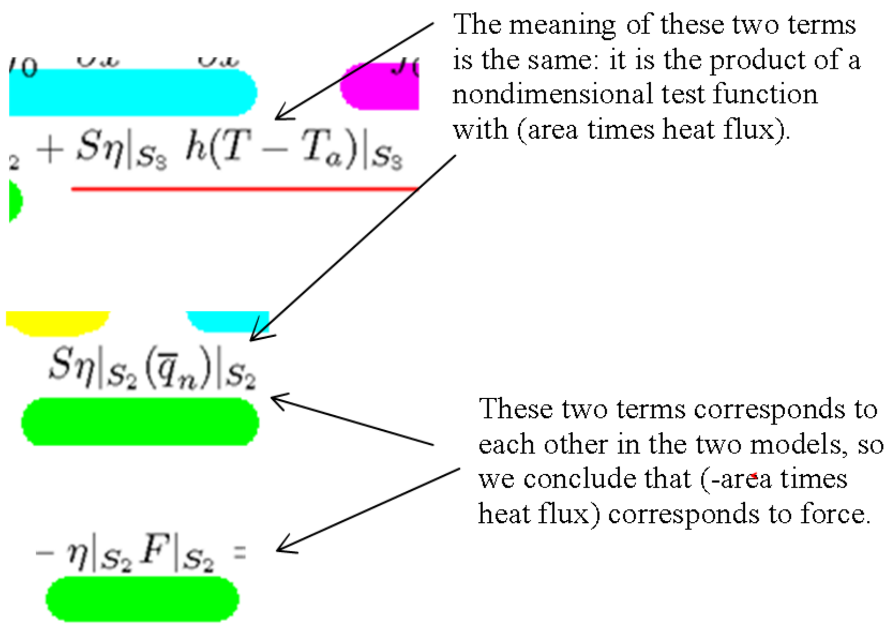
Heat flux BC ~ Force BC

Temperature ~ Deflection
Temperature gradient ~ Slope
Heat flux ~ Transverse force

Clearly one term from the heat conduction problem was not matched

2

Panel 3



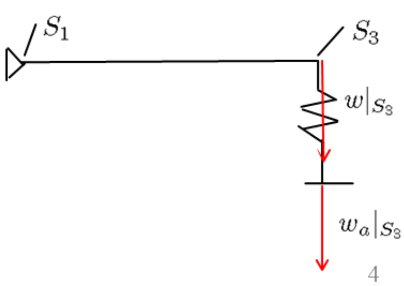
Panel 4

Therefore, we must conclude that $S\, h(T - T_a)|_{S_3}$ must correspond to a force in the boundary condition we are searching for for the cable model.

Furthermore, we know that in the two models we have the correspondence of temperature and deflection. Therefore, $S\, h$ corresponds to a spring constant.

Thus we finally conclude that the boundary conditions we are looking for is a spring support:

Force in the spring= $F|_{S_3} = -k(w - w_a)|_{S_3}$



4

Panel 5

The weighted residual equation for the cable model including the spring-support boundary condition may be written as

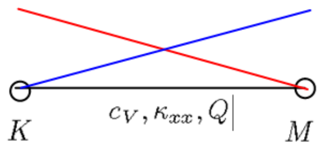
$$\int_0^L \eta\mu\ddot{w}\, dx + \int_0^L \frac{\partial\eta}{\partial x} P \frac{\partial w}{\partial x}\, dx - \int_0^L \eta q\, dx - \eta|_{S_2} F|_{S_2} + \eta|_{S_3} k(w - w_a)|_{S_3} = 0$$

5

Panel 1

Exercise 59-d

Formulate the finite element expressions for the Galerkin model of heat conduction with one spatial coordinate using the L2 finite elements.



1

Panel 2

In exercise 59-a we have formulated the Galerkin model of heat conduction using one spatial coordinate:

$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\,dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\,dx - S\int_0^L \eta Q\,dx +$$

$$S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\,h(T-T_a)|_{S_3} = 0, \qquad \eta|_{S_1} = 0$$

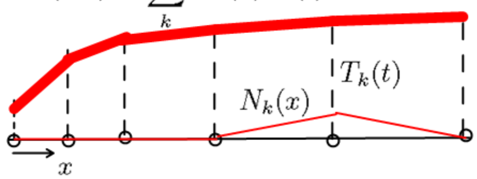
In order to develop the finite element formulation, we shall start with writing down the finite element expansions of the trial function and the test functions.

2

Panel 3

The trial function describes the variation of the temperature along the thickness of the wall as a linear combination of though finite element basis functions where the coefficients of the linear combination are functions of time. (Exactly parallel to the cable model.)

$$T(x,t) = \sum_k N_k(x) T_k(t)$$



The test functions are all those basis functions which vanish at the point where the essential boundary conditions are prescribed

$$N_k(x)|_{S_1} = 0$$

which practically translates into all basis functions except those at nodes where the temperature is prescribed.

3

Panel 4

Substituting the finite element expansions for the trial function and the test function leads to the analogy of (2.15) for the cable model:

$$S\int_0^L N_j(x) c_V \sum_k N_k(x)\dot{T}_k(t)\, dx + S\int_0^L N_j'(x)\kappa_{xx} \sum_k N_k'(x) T_k(t)\, dx$$

$$-S\int_0^L N_j(x) Q\, dx + SN_j(x)|_{S_2}(\overline{q}_n)|_{S_2} + SN_j(x)|_{S_3}\; h(\sum_k N_k(x)T_k(t) - T_a)|_{S_3} = 0$$

$$N_j(x)|_{S_1} = 0$$

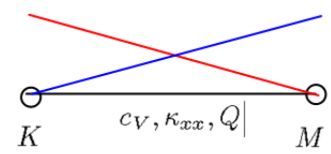
The resulting matrix equations consist of (in the order given above): capacity matrix times rates of temperatures, conductivity matrix times temperatures, load due to heat generation, load due to applied heat flux, load due to surface heat transfer.

4

Panel 5

We can also note that the same assembly techniques used for the cable (and in fact for all the finite element methods discussed in this book) are applicable: compute elementswise matrices and vectors, and then assemble them into the global matrices/vectors.

Capacity elementwise matrix

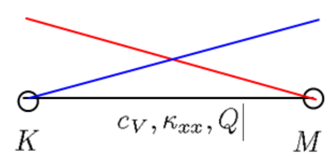


$$(\boldsymbol{C}^{(e)})_{KM} = S\int_0^L N_K(x) c_V N_M(x)\, dx$$

5

Panel 6

Conductivity elementwise matrix



$$(\boldsymbol{K}^{(e)})_{KM} = S\int_0^L N'_K(x) \kappa_{xx} N'_M(x)\, dx$$

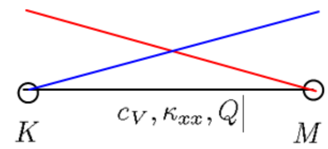
Elementwise heat-generation loads

$$(\boldsymbol{L}^{(e)})_{K} = S\int_0^L N_K(x) Q\, dx$$

6

Panel 7

For applied heat flux



add load

$$(\boldsymbol{L})_j = -S(\overline{q}_n)|_{S_2}$$

where node j is located on the boundary S_2 with prescribed heat flux

For heat surface-transfer boundary condition

add load

$$(\boldsymbol{L})_j = ShT_a|_{S_3}$$

where node j is located on the boundary S_3 with prescribed heat surface-transfer boundary condition

add to the heat-surface-transfer matrix

$$(\boldsymbol{H})_{jj} = -Sh$$

7

Panel 8

When all the element contributions (and the contributions from the boundary conditions) are assembled, the resulting system of linear differential equations reads

$$\sum_{\text{free } i} C_{ji}\dot{T}_j + \sum_{\text{free } i} K_{ji}T_i + \sum_{\text{free } i} H_{ji}T_i = L_j$$

8

Panel 1

Exercise 59-e

Develop the view of the spring-support boundary condition as a penalty enforcement of the prescribed displacement in the Galerkin model for the vibrating cable.

1

Panel 2

In exercise 59-b we have compared the two formulations term by term:

$$S\int_0^L \eta c_V \frac{\partial T}{\partial t}\,dx + S\int_0^L \frac{\partial \eta}{\partial x}\kappa_{xx}\frac{\partial T}{\partial x}\,dx - S\int_0^L \eta Q\,dx +$$
$$S\eta|_{S_2}(\overline{q}_n)|_{S_2} + S\eta|_{S_3}\,h(T-T_a)|_{S_3} = 0, \quad \eta|_{S_1} = 0$$

$$\int_0^L \eta\mu\ddot{w}\,dx + \int_0^L \frac{\partial \eta}{\partial x}P\frac{\partial w}{\partial x}\,dx - \int_0^L \eta q\,dx - \eta|_{S_2}F|_{S_2} = 0,$$
$$\eta|_{S_1} = 0$$

Capacity ~ Mass

Conductivity ~ Stiffness

Heat generation ~ Transverse load

Heat flux BC ~ Force BC

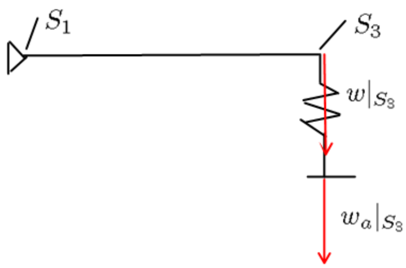
Temperature ~ Deflection
Temperature gradient ~ Slope
Heat flux ~ Transverse force

Clearly one term from the heat conduction problem was not matched

2

Panel 3

Force in the spring= $F|_{S_3} = -k(w - w_a)|_{S_3}$



3

Panel 4

The weighted residual equation for the cable model including the spring-support boundary condition may be written as

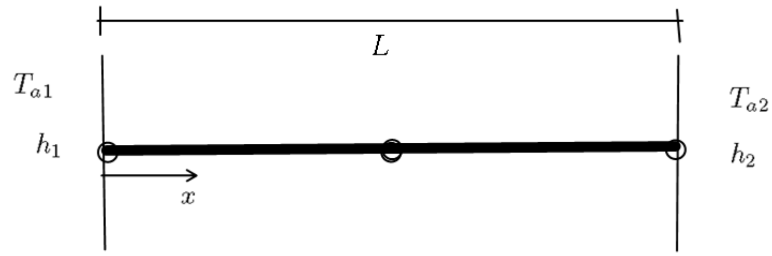
$$\int_0^L \eta\mu\ddot{w}\,dx + \int_0^L \frac{\partial\eta}{\partial x} P \frac{\partial w}{\partial x}\,dx - \int_0^L \eta q\,dx - \eta|_{S_2} F|_{S_2} + \eta|_{S_3} k(w - w_a)|_{S_3} = 0$$

4

Panel 1

Exercise 60-a

Solve the steady-state heat conduction problem below with a one-coordinate Galerkin model using two L2 finite elements. The ambient temperature is given on either side of a homogeneous wall. The heat-surface transfer coefficients are different on the two faces.



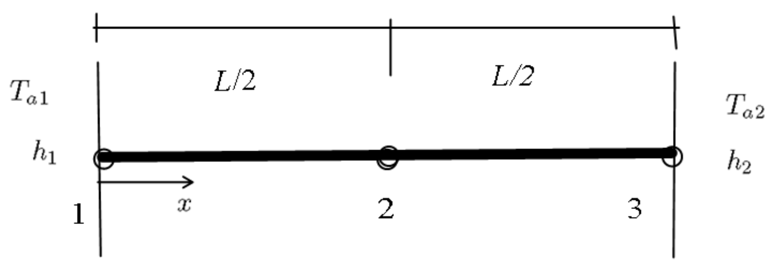
1

Panel 2

Finite element mesh:

Elements	Nodes
1	1,2
2	2,3

Equation #'s = Node #'s



2

Panel 3

The weighted residual simplifies in this case to

$$S \int_0^L N_j'(x)\kappa_{xx} \sum_k N_k'(x) T_k(t)\, dx + S N_j(x)|_{S_3}\ h(\sum_k N_k(x)T_k(t) - T_a)|_{S_3} = 0$$

Note that S_3 consists in this case of two points, *x=0*, and x=*L*.

The elementwise conductivity matrix can in the present case be evaluated (analytically) as

$$\boldsymbol{K}^{(e)} = \frac{S\kappa_{xx}}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad h = L/2$$

It is worthwhile to note the resemblance to the stiffness matrices for the cable elements, the only thing that changed are the constants in front.

3

Panel 4

The global conductivity matrix is assembled from the two elementwise conductivity matrices as

$$\boldsymbol{K} = \frac{2S\kappa_{xx}}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Note that this matrix is singular:

```
>> rank([1,-1,0;-1,2,-1;0,-1,1])

ans =

    2
```

4

Panel 5

As outlined in exercise 59-d, for the heat surface-transfer boundary condition terms needs to be added to both the heat load and to the heat surface transfer matrix.

add load

$$(\boldsymbol{L})_j = ShT_a|_{S_3}$$

add to the heat-surface-transfer matrix

$$(\boldsymbol{H})_{jj} = -Sh$$

The heat surface transfer matrix is then found as

$$\boldsymbol{H} = \begin{pmatrix} -Sh_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -Sh_2 \end{pmatrix}$$

and the heat surface transfer load is

$$\boldsymbol{L} = \begin{pmatrix} Sh_1T_{a1} \\ 0 \\ Sh_2T_{a2} \end{pmatrix}$$

5

Panel 6

The solution is easily found with Matlab's symbolic algebra:

$\frac{2S\kappa_{xx}}{L}$ Sh_1 Sh_2 T_{a1} T_{a2}

```
syms k s1 s2 ta1 ta2 real

K=k*[1,-1,0;-1,2,-1;0,-1,1]
H=[-s1,0,0;0,0,0;0,0,-s2]
L= [s1*ta1,0,s2*ta2]'
(K+H)\L
ans =

       -(k*s1*ta1+k*s2*ta2-2*s2*s1*ta1)/(k*s1+k*s2-2*s1*s2)
 -(k*s1*ta1+k*s2*ta2-s1*s2*ta2-s2*s1*ta1)/(k*s1+k*s2-2*s1*s2)
       -(k*s1*ta1+k*s2*ta2-2*s1*s2*ta2)/(k*s1+k*s2-2*s1*s2)
```

This may be simplified to give

$$T_1 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - 2 * ta1)}{k/s2 + k/s1 - 2}$$

$$T_2 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - ta2 - ta1)}{k/s2 + k/s1 - 2}$$

$$T_3 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - 2 * ta2)}{k/s2 + k/s1 - 2}$$

6

Panel 7

It may be instructive to consider the solution for the heat surface transfer coefficients very large (much larger than the conductivity coefficient):

The expressions

$$T_1 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - 2 * ta1)}{k/s2 + k/s1 - 2}$$

$$T_2 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - ta2 - ta1)}{k/s2 + k/s1 - 2}$$

$$T_3 = \frac{-(k/s2 * ta1 + k/s1 * ta2 - 2 * ta2)}{k/s2 + k/s1 - 2}$$

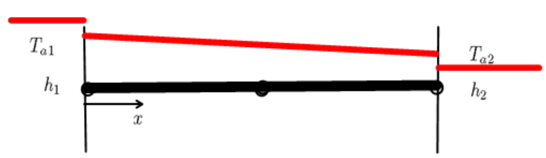
would tend to

$$T_1 = T_{a1}, T_2 = (T_{a1} + T_{a2})/2, T_3 = T_{a2}$$

7

Panel 8

For finite values of the heat surface transfer coefficients, the distribution of temperature would in general look like



Note the jumps in the temperature at the surfaces of the wall: the larger the heat surface transfer coefficient, the smaller the jump.

For vanishingly small heat surface transfer coefficients, $h_1 \to 0, h_2 \to 0$

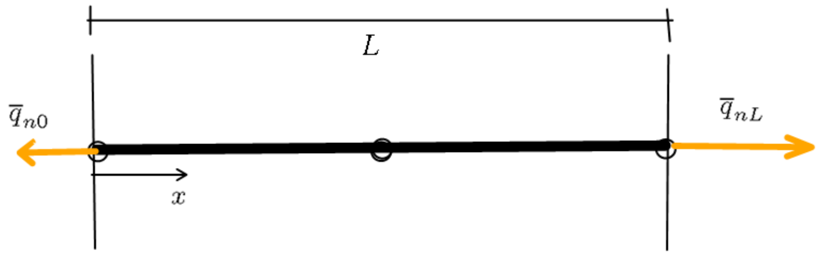
the solution for the temperatures will cease to have a unique solution: the matrix H will become a zero matrix, and $K+H$ will be singular.

8

Panel 1

Exercise 60-b

Solve the steady-state heat conduction problem below with a one-coordinate Galerkin model using two L2 finite elements. The wall is loaded by heat fluxes on either side.



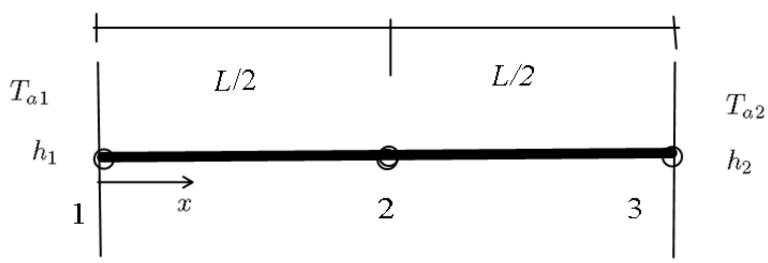
1

Panel 2

Finite element mesh:

Elements	Nodes
1	1,2
2	2,3

Equation #'s = Node #'s



2

Panel 3

The weighted residual simplifies in this case to

$$S\int_0^L N_j'(x)\kappa_{xx}\sum_k N_k'(x)T_k(t)\,dx + SN_j(x)|_{S_2}(\overline{q}_n)|_{S_2} = 0$$

Note that S_2 consists in this case of two points, x=*0*, and x=L.

The elementwise conductivity matrix can in the present case be evaluated (analytically) as

$$\boldsymbol{K}^{(e)} = \frac{S\kappa_{xx}}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad h = L/2$$

3

Panel 4

The global conductivity matrix is assembled from the two elementwise conductivity matrices as

$$\boldsymbol{K} = \frac{2S\kappa_{xx}}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Note that this matrix is singular:

```
>> rank([1,-1,0;-1,2,-1;0,-1,1])

ans =

     2
```

4

Panel 5

The heat loads are assembled as follows:

The term $SN_j(x)|_{S_2}(\overline{q}_n)|_{S_2}$ needs to be evaluated at two points, $x=0$, and $x=L$. Only one basis function is nonzero at either point. Hence,

$$F_1 = -SN_1(x=0)\overline{q}_{n0} = -S\overline{q}_{n0}$$

and

$$F_3 = -SN_3(x=L)\overline{q}_{nL} = -S\overline{q}_{nL}$$

The global load vector is therefore

$$\boldsymbol{L} = \begin{pmatrix} -S\overline{q}_{n0} \\ 0 \\ -S\overline{q}_{nL} \end{pmatrix}$$

5

Panel 6

Thus, the system of linear equations to be solved is

$$\frac{2S\kappa_{xx}}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} -S\overline{q}_{n0} \\ 0 \\ -S\overline{q}_{nL} \end{pmatrix}$$

We have noted before that the system matrix was singular. Does a solution to this system exist under these conditions?

First, we make use of the fact that the second equation may be written as

$$(-T_1 + T_2) + (T_2 - T_3) = 0$$

We see these differences (with opposite signs) in the first and last equation. Substituting, we obtain that both the first and the last equations may be true (and therefore a solution does exist) provided

$$\overline{q}_{n0} = -\overline{q}_{nL}$$

6

Panel 7

The physical explanation is that the weighted residual expresses a balance of heat energy. Since there are no heat loads generated inside the wall, in the steady state what ever amount of energy enters the wall at one face must leave the wall at the other face.

Therefore, the condition $\overline{q}_{n0} = -\overline{q}_{nL}$

simply states that no heat energy accumulates inside the wall. Under these conditions a distribution of temperature exists to support such state of affairs.

However, we see that the distribution of temperature is definitely not determined uniquely by the equations. We show that easily by recognizing that the system matrix is singular and therefore a nonzero solution exists for zero right hand side:

$$\boldsymbol{K}\tilde{\boldsymbol{T}} = \boldsymbol{0}$$

Another way of saying this is by writing

$$\boldsymbol{K}\tilde{\boldsymbol{T}} = 0 \times \tilde{\boldsymbol{T}}$$

7

Panel 8

The equation

$$\boldsymbol{K}\tilde{\boldsymbol{T}} = 0 \times \tilde{\boldsymbol{T}}$$

is of course an eigenvalue problem. We know that the two statements "the matrix is singular" and "the matrix has eigenvalue zero" are equivalent.

So we know that the system leads to a nonzero solution

$$\boldsymbol{K}\boldsymbol{T} = \boldsymbol{L}$$

provided the right-hand side is of the form

$$\boldsymbol{L} = \begin{pmatrix} -S\overline{q}_{n0} \\ 0 \\ +S\overline{q}_{n0} \end{pmatrix}$$

8

Panel 9

We can add together the two equations

and $$KT = L$$

so that we see that $$K\tilde{T} = 0$$

is also a solution: $$K(T + \tilde{T}) = L$$

This explains our claim that the solution was not unique.

The Matlab eig() function can illustrate these observations:

```
>> [v,d] =eig([1,-1,0;-1,2,-1;0,-1,1])

v =
 -0.577350269189626  -0.707106781186547   0.408248290463863
 -0.577350269189626   0.000000000000000  -0.816496580927726
 -0.577350269189626   0.707106781186547   0.408248290463863

d =
  0.000000000000000                   0                   0
                  0   1.000000000000000                   0
                  0                   0   3.000000000000000
```

Eigenvectors

Eigenvalues

9

Panel 10

The first eigenvalue is equal to zero. The eigenvector gives the components of $\tilde{T}$ and we see that the corresponding solution is "uniform temperature".

The second eigenvector corresponds to applied heat fluxes at $x=0$, and $x=L$. The third eigenvector is not useful, because it would correspond to a situation in which heat flux would be applied at the interior node 2: this is not a realistic scenario.

10